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Injectives in varieties of groups

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The purpose of this note is to correct some errors of the authors recorded on page 144 of Hanna Neumann's book [2].

Let $\mathbf{V}$ be a variety of groups, and $G \in \mathbf{V}$. Say that G is $\mathbf{V}$-*injective* if $\mathbf{V} \ni A \geqslant B \xrightarrow{\beta} G$ implies the existence of a homomorphism $\alpha : A \to G$ which agrees on B with β. Call G *slightly* $\mathbf{V}$-*injective* if such α exist whenever β is one-to-one, and $\mathbf{V}$-*closed* if the existence of α is required only when β is an isomorphism. Let G be *relatively injective* (*relatively slightly injective, relatively closed*) if G is $\mathbf{V}$-injective (slightly $\mathbf{V}$-injective, $\mathbf{V}$-closed) for at least one choice of $\mathbf{V}$ (so in particular when $\mathbf{V}$ is the variety generated by G).

It was claimed on page 144 of [2] that a directly indecomposable finite group is relatively injective if and only if it is an extension of a cyclic group of prime power order by an automorphism of coprime order. In that statement, "relatively injective" should be replaced by "relatively slightly injective". (In fact, we prove at the end of this note that all relatively injective groups are abelian.) The results reported on that page concerning relatively closed groups still appear to be correct.

Consider more closely the case $\mathbf{V} = \mathbf{A}_3\mathbf{A}_2$. The reference to a partial analogue in this variety for the standard theory of injectives was envisaged in terms of $\mathbf{A}_3\mathbf{A}_2$-closed groups. A finite group is $\mathbf{A}_3\mathbf{A}_2$-closed if and only if it is a direct product of groups of order 2 and of copies of the symmetric group S_3. We have no similarly conclusive result for infinite groups, but there are certainly "enough" $\mathbf{A}_3\mathbf{A}_2$-closed groups: the unrestricted direct powers of S_3 are $\mathbf{A}_3\mathbf{A}_2$-closed, and embed every group of this variety. Let a, b be elements in S_3 of orders 2 and 3, respectively, and B the subgroup of the direct product $S_3 \times S_3$ generated by (a, a), $(b, 1)$ and $(1, b)$. It is easy to see that there is a one-to-one homomorphism $\beta : B \to S_3 \times S_3 \times S_3$ such that $(a, a)\beta = (a, a, a)$, $(b, 1)\beta = (v, b, 1)$, $(1, b)\beta = (1, b, b)$ and that no proper subgroup of $S_3 \times S_3$ containing B, nor any proper

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subgroup of $S_3 \times S_3 \times S_3$ containing $B\beta$, is $\mathbf{A}_3\mathbf{A}_2$-closed. Thus there is no real analogue for injective hulls in this context. One can further see that β cannot be extended to a homomorphism $\alpha: S_3 \times S_3 \to S_3 \times S_3 \times S_3$ (for instance, by using that B is normal in $S_3 \times S_3$ but $B\beta$ is its own normalizer in $S_3 \times S_3 \times S_3$). Thus while S_3 is slightly $\mathbf{A}_3\mathbf{A}_2$-injective, $S_3 \times S_3 \times S_3$ is not. With other nonabelian choices of $\mathbf{V}$, the results we reached were at least as negative as these. Thus neither the relatively closed nor the relatively slightly injective case seems interesting enough to justify the space which would be required to elaborate it.

On the other hand, the relatively injective case can be dealt with briefly and completely. In a forthcoming paper [1], O. C. Garcia and F. Larrión determine all abelian $\mathbf{V}$-injectives, for each choice of $\mathbf{V}$. We conclude this note by proving the following.

THEOREM. *All relatively injective groups are abelian.*

It is convenient to establish first that if H is an abelian subgroup of a relatively injective group G, then G has an endomorphism γ which leaves each element of H fixed and whose image $G\gamma$ centralizes H. To this end, let $A = G \times G$, $B = H \times H$, $\beta: (h, h') \mapsto hh'$, and $\alpha: A \to G$ a homomorphism which agrees with β on B, so that in particular

$$(h, 1)\alpha = (1, h)\alpha = h \quad \text{whenever} \quad h \in H.$$

Define γ by $\gamma: g \mapsto (g, 1)\alpha$; this is clearly an endomorphism of G which acts identically on H. Also,

$$(g\gamma)h = (g, 1)\alpha(1, h)\alpha = [(g, 1)(1, h)]\alpha = [(1, h)(g, 1)]\alpha = h(g\gamma)$$

whenever $g \in G$ and $h \in H$.

We apply this preliminary result twice. Let x, y be arbitrary elements of G. Consider first a γ corresponding to the abelian subgroup generated by the commutator $[x, y]$: then

$$[x\gamma, y\gamma] = [x, y]\gamma = [x, y] \tag{1}$$

and the subgroup generated by $x\gamma$ and $[x\gamma, y\gamma]$ is abelian. Second, let δ be an endomorphism corresponding to this subgroup: then $G\delta$ centralizes $x\gamma$, so $[x\gamma, y\gamma\delta] = 1$, and δ leaves $x\gamma$ and $[x\gamma, y\gamma]$ fixed, so

$$1 = [x\gamma, y\gamma\delta] = [x\gamma\delta, y\gamma\delta] = [x\gamma, y\gamma]\delta = [x\gamma, y\gamma].$$

In view of (1) this proves that $[x, y] = 1$, and our argument is complete.

REFERENCES

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[2] HANNA NEUMANN, *Varieties of groups* (Ergebnisse der Mathematik und ihrer Grenzgebiete, 37. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

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