Maximal Frattini extensions

By

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1. Motivation. At the recent Santa Cruz conference on finite groups, several participants asked variants of the following question. Given a finite group G, how large a finite group A can be extended by G so that A falls into the Frattini subgroup of the extension? (Of course, as Frattini subgroups of finite groups are nilpotent, only nilpotent A come into consideration.) A moment's thought shows that when G has prime order, there are such A of arbitrarily large order, and it is easy to see that the same is true in general (provided only that G is not trivial). So the real question is to identify just what measure of the size of A is relevant here. The answer: the cardinalities of the minimal generating sets of the Sylow subgroups, *and nothing else*. A related question was investigated by Gaschütz over twenty-five years ago, and this work was inspired by his results.

2. Discussion. To have a convenient language, let *Frattini extension* mean a surjective group homomorphism whose kernel is contained in the Frattini subgroup of its domain. The first step is to observe the following simple fact.

2.1. *If $\gamma: H \to G$ is a Frattini extension of finite groups with G of order n, say, then the kernel A of γ can be generated by $n^2 - n + 1$ elements.*

(Proofs of all displayed statements are deferred to later sections.) Since A is nilpotent, all Sylow subgroups of A can also be generated by $n^2 - n + 1$ elements. For each prime p, let $\Phi(G, p)$ denote the maximum of the cardinalities of the minimal generating sets of the Sylow p-subgroups of all such A. The second half of the answer claims that A can be arbitrarily large subject *only* to the restriction expressed by the $\Phi(G, p)$. One way to make this precise is to take an arbitrary variety $\mathfrak{V}$ of locally finite groups, write $\mathfrak{V}_p$ for the variety consisting of the p-groups in $\mathfrak{V}$, and consider all Frattini extensions $\gamma: H \to G$ with kernel A in $\mathfrak{V}$ and G fixed. By 2.1, the orders of these A are bounded by the (finite) order of the $\mathfrak{V}$-free group of rank $n^2 - n + 1$, so it is possible to choose γ $\mathfrak{V}$-maximal in the sense that, for G fixed, the order of H is maximal subject to $A \in \mathfrak{V}$.

2.2. *If $\gamma: H \to G$ is $\mathfrak{V}$-maximal, then the Sylow p-subgroup of its kernel A is $\mathfrak{V}_p$-free of rank $\Phi(G, p)$.*

In addition to providing an answer to the Santa Cruz question, these extensions have even more interesting properties.

2.3. *If $\gamma: H \to G$ is $\mathfrak{V}$-maximal, $\alpha: G \to D$ is a surjective homomorphism, and $\delta: C \to D$ is any Frattini extension with kernel in $\mathfrak{V}$, then there exist surjective $\beta: H \to C$ such that $\beta\delta = \gamma\alpha$.*

In particular, take $D = G$, let α be the identity automorphism ι of G, and δ also $\mathfrak{V}$-maximal. Then 2.3 can also be applied with the roles of γ and δ interchanged; hence C and H have the same order, and so the surjective β of 2.3 must be isomorphisms. In this rather strong sense, the $\mathfrak{V}$-maximal γ is determined by G (and $\mathfrak{V}$) up to isomorphism.

Still with $D = G$ and $\alpha = \iota$, one might paraphrase this special case of 2.3 as follows: the Frattini extensions of $\mathfrak{V}$-groups by G are precisely the quotients of γ.

With $\delta = \gamma$, one gets that each automorphism α of G lifts to an automorphism β of H. Here, however, a warning is called for: in general, one *cannot* choose one β for each α in a coherent, functorial manner; that is, so that $\alpha = \alpha_1\alpha_2$ would imply $\beta = \beta_1\beta_2$. (For instance, let G be a noncyclic group of order p^2 and $\mathfrak{V}$ the variety $\mathfrak{A}_p$ of all elementary abelian p-groups. Then the largest abelian quotient H/H' of H is the direct product of two cyclic groups, each of order p^2. Restriction provides a homomorphism of the automorphism group of H/H' onto that of G, but when $p > 3$ the proof of 4.2.2 in Wall [9] is readily adaptable to show that this homomorphism does not split.

The uniqueness of $\mathfrak{A}_p$-maximal Frattini extensions was established long ago by Gaschütz [2]. In that case, of course, the relatively free nature of the kernels is not an issue. If $\gamma: H \to G$ is such an extension with kernel A, one may view A as a G-module over the field $\mathbb{F}_p$ of p elements; by 2.2, the dimension of A is $\Phi(G, p)$. Gaschütz [2] gave two characterizations of A as $\mathbb{F}_pG$-module; these are still the only means available for actually calculating $\Phi(G, p)$. First, if $G = F/R$ with F free of finite rank, and the largest elementary abelian p-quotient $R/R'R^p$ of R is regarded as an $\mathbb{F}_pG$-module in the natural way, then the quotient of $R/R'R^p$ modulo any maximal $\mathbb{F}_pG$-projective submodule is isomorphic to A. Second, if

$$\cdots \to P_2 \to P_1 \to \mathbb{F}_p \to 0$$

is any minimal projective resolution (in the category of $\mathbb{F}_pG$-modules) of the trivial module $\mathbb{F}_p$, then the kernel of $P_2 \to P_1$ is isomorphic to A. Consequently, $\Phi(G, p) = 0$ if and only if p does not divide the order of G.

Further properties of this kernel were discussed recently by Griess and Schmid in [3]. In particular, they showed that the centralizer of A in G is precisely (the largest normal p'-subgroup) $0_{p'}(G)$, unless G is p-soluble with non-trivial cyclic Sylow p-subgroups in which case A is isomorphic to each chief p-factor of G. Thus $\Phi(G, p) = 1$ precisely in this exceptional case.

(When $\Phi(G, p) > 1$ and U is any $\mathbb{F}_p G$-module on which $0_{p'}(G)$ acts trivially, an easy application of the results of Bryant and Kovács [1] shows that $\mathfrak{V}$ can be chosen large enough to ensure that the centre of the kernel of a $\mathfrak{V}$-maximal Frattini extension by G contain a G-isomorphic copy of U. At least in this weak sense, the kernels can be chosen "arbitrarily large" not only as abstract groups but also in terms of the "action" of G).

The proofs of 2.1—2.3 are elementary, almost every step being a familiar piece of general nonsense (though we shall not present them in their abstract setting). Still, they did not all come to mind when the questions were raised at Santa Cruz. Initial responses relied on repeated application of the Gaschütz results on $\mathfrak{A}_p$-maximal Frattini extensions, and the key step from that argument may retain some interest. Let $\gamma\colon H \to G$ and $\varkappa\colon K \to H$ be $\mathfrak{A}_p$-maximal Frattini extensions, with kernels A and B, respectively; write C for the kernel of $\varkappa\gamma$, and let $\varkappa_*\colon C \to A$ be the restriction of $\varkappa$. Argue that $\varkappa\gamma$ restricted to the factor group of K modulo the Frattini subgroup of C is still a Frattini extension, so by the $\mathfrak{A}_p$-maximality of γ the Frattini subgroup of C must be precisely B. Thus $\varkappa_*$ is a Frattini extension. For *any* subgroup A of H, the fact that projective H-modules restricted to A remain projective, together with either Gaschütz characterization above, implies that the kernel of the $\mathfrak{A}_p$-maximal Frattini extension by A must be a direct summand of B. Hence $\varkappa_*$ *is* $\mathfrak{A}_p$-maximal; and as A is elementary abelian, one can now readily identify C as a free group of the product variety $\mathfrak{A}_p \mathfrak{A}_p$. It is routine to extend this to a proof of 2.2, but the process uses more additional general nonsense than the whole of the proof we give in Section 3 (based on an idea of P. Hall).

The arguments can be made even shorter, and the results more pleasing, if one is prepared to use the power of the much more sophisticated context of profinite groups. There one may consider Frattini extensions which are continuous homomorphisms of profinite groups, the appropriate definition of Frattini subgroup being the intersection of the maximal open subgroups: see Gruenberg [4]. A Frattini extension of profinite groups will be called projective if its domain is projective (with respect to surjective maps in the concrete category of profinite groups and continuous homomorphisms). With "$\mathfrak{V}$-maximal" replaced by "projective", the analogues of 2.3 and of its consequences may be proved simply by adapting the short proof of Proposition 2 of Gruenberg [4]. The only issue is the existence of enough projective Frattini extensions.

2.4. *To each profinite group G, there exists a Frattini extension $\pi\colon P \to G$ with projective P.*

(When G is finite, the $\mathfrak{V}$-maximal $\gamma\colon H \to G$ is obtained from this π modulo the $\mathfrak{V}$-verbal subgroup of $\ker \pi$.) The proof of 2.4 rests on the following result, which is implicit in Gruenberg [4] (compare Corollaire 3 on page I-37 of Serre [8]).

2.5. *Closed subgroups of projective profinite groups are projective.*

By Theorem 2 of Gruenberg [4], projective pro-p-groups are restricted free pro-p-groups. Thus 2.4 and 2.5 yield also that the Sylow p-subgroup A_p of the (pro-nilpotent) kernel of π is a restricted free pro-p-group. The Sylow p-subgroups of any projective profinite group P are restricted free pro-p-groups, hence P is torsion-free as an abstract group. Though it can be done, there is little point in elaborating an *a priori* definition of $\Phi(G, p)$ for general profinite G. However, to see that we really have a generalization of 2.2, we must note that when G is finite the restricted free rank of A_p is actually $\Phi(G, p)$ as defined before.

Gruenberg's Theorem 2 is a direct consequence of his simple Proposition 2, given that duality provides a profinite version of the fact that all discrete elementary abelian p-groups are restricted direct powers of the group of order p. By contrast, 2.5 lies deeper: it appears to depend on Gruenberg's characterization of projectives as groups with cohomological dimension at most 1. In the last section of this paper we show that 2.5 may also be derived directly, by elementary wreath product techniques instead of cohomology.

(For finite G, the profinite results were originally derived also by repeated use of $\mathfrak{A}_p$-maximal Frattini extensions. In that argument, we needed an observation which may be of independent interest: if $\gamma\colon H \to G$ is an $\mathfrak{A}_p$-maximal Frattini extension of finite groups, then all elements of order p in H lie in the kernel of γ.)

We are indebted to Professor Gruenberg for long and enlightening discussions.

3. The finite case. Three familiar facts about finite Frattini extensions will be used repeatedly.

3.1. *If $\varphi\colon L \to H$ is a surjective homomorphism of finite groups and M is minimal among the subgroups of L with $M\varphi = H$, then the restriction $\varphi_M\colon M \to H$ is a Frattini extension.*

For if a maximal subgroup of M failed to contain $\ker \varphi_M$, it would supplement $\ker \varphi_M$ in M and hence also $\ker \varphi$ in L, contrary to the minimality of M.

3.2. *If $\gamma\colon H \to G$ is a finite Frattini extension and $\varphi\colon F \to H$ is a homomorphism such that $\varphi\gamma$ is surjective, then φ is also surjective.*

For $F\varphi$ supplements the kernel of γ and hence also Frat H, but Frattini subgroups of finite groups have no proper supplements.

3.3. *Composites of Frattini extensions are Frattini extensions.*

Let $\mu\colon M \to H$ and $\gamma\colon H \to G$ be Frattini extensions. As $\ker \mu \leqq \operatorname{Frat} M$, we have $(\operatorname{Frat} M)\mu = \operatorname{Frat} H$: thus Frat M is the complete inverse image $(\operatorname{Frat} H)\mu^{-1}$. Now $\ker \mu\gamma = (\ker \gamma)\mu^{-1} \leqq (\operatorname{Frat} H)\mu^{-1} = \operatorname{Frat} M$ as required.

Proof of 2.1. Let ϱ be a homomorphism of a free group F of rank n onto G. By the projective property of free groups, there is a $\varphi\colon F \to H$ with $\varphi\gamma = \varrho$. Now 3.2 shows that $F\varphi = H$ and so $(\ker \varrho)\varphi = \ker \gamma$. According to Schreier's Theorem, the kernel $\ker \varrho$ is a free group of rank $1 + n\,(n - 1)$. ■

3.4. *If $\gamma: H \to G$ is a $\mathfrak{V}$-maximal Frattini extension, then its kernel A is $\mathfrak{V}$-projective.*

By this we mean that A is projective with respect to surjective maps in the concrete category whose objects are the groups of $\mathfrak{V}$ and whose maps are the homomorphisms between these groups. This is essentially the terminology of Hanna Neumann's [6], which the reader may wish to consult for all basic facts involving varieties. (Originally, P. Hall [5] used the term "splitting groups in $\mathfrak{V}$". Note that in [6] "epimorphism" means "surjective homomorphism". If epimorphisms are defined by the now usual cancellation property, $\mathfrak{V}$-projectives with respect to epimorphisms will, for some $\mathfrak{V}$, form a smaller class than $\mathfrak{V}$-projectives with respect to surjectives, but this never happens for soluble $\mathfrak{V}$: see Peter M. Neumann [7]. In the present context we could, of course, restrict attention even to nilpotent $\mathfrak{V}$ without any real loss.) Standard arguments yield that all $\mathfrak{V}$-free groups are $\mathfrak{V}$-projective, and that all retracts (that is, subgroups with normal complements) of $\mathfrak{V}$-projectives are $\mathfrak{V}$-projective.

Proof of 3.4. Choose ϱ and φ as in the proof of 2.1; put $\ker \varrho = R$ and $\ker \varphi = S$. Now $R/S \cong A \in \mathfrak{V}$, so S contains the verbal subgroup V of R corresponding to $\mathfrak{V}$. Since R is free, R/V is $\mathfrak{V}$-free. Choose a subgroup M/V in F/V minimal with respect to $M\varphi = H$. As in 3.1, the restriction $\varphi_{M/V}: M/V \to H$ of φ is a Frattini extension. By 3.3, $\varphi_{M/V}\gamma: M/V \to G$ is also a Frattini extension; its kernel is, of course, $(R \cap M)/V$. However, $(R \cap M)/V \leqq R/V \in \mathfrak{V}$ and γ is $\mathfrak{V}$-maximal, so the kernel $(S \cap M)/V$ of $\varphi_{M/V}$ must be trivial. In other words $S \cap M = V$, and therefore $A \cong R/S \cong (R \cap M)/V$ shows that A is isomorphic to a retract of R/V. ■

Proof of 2.2. It follows from 3.4 that the Sylow p-subgroup A_p of A is also $\mathfrak{V}$-projective, being a retract of the (nilpotent) $\mathfrak{V}$-projective A. Let P be a $\mathfrak{V}_p$-free group whose rank is the cardinality of a minimal generating set of A_p. Then A_p and P have isomorphic Frattini quotients, so the $\mathfrak{V}_p$-projective property of P yields a Frattini extension $P \to A_p$. Since A_p is $\mathfrak{V}$-projective, this must be an isomorphism: thus A_p is $\mathfrak{V}_p$-free. (This is essentially how Hall showed in [5] that $\mathfrak{V}_p$-projectives are $\mathfrak{V}_p$-free; the argument is reproduced here because it fits well into the present context but is embedded in more complex material both in Hall's [5] and in Hanna Neumann's text [6].) In view of 2.3, what remains is to show that the $\mathfrak{V}_p$-free rank of A_p is independent of $\mathfrak{V}$. When $\mathfrak{V}$ contains no nontrivial p-groups, this is merely a matter of interpretation: in that case, all free groups of $\mathfrak{V}_p$ are trivial. Otherwise $\mathfrak{V}_p$ contains $\mathfrak{A}_p$, so by 2.3 (applied twice) the largest $\mathfrak{A}_p$-quotient of A is the kernel of an $\mathfrak{A}_p$-maximal Frattini extension by G, and of course that quotient is isomorphic to the Frattini factor group of A_p. ■

Proof of 2.3. Consider the subdirect product L consisting of the elements (h, c) of the direct product $H \times C$ such that $h\gamma\alpha = c\delta$, and the homomorphisms $\varphi: L \to H$ and $\psi: L \to C$ defined by $(h, c)\varphi = h$, $(h, c)\psi = c$, so $\varphi\gamma\alpha = \psi\delta$. Check that φ and ψ are surjective. Choose a subgroup M in L minimal with respect to $M\varphi = H$: by 3.1, the restriction φ_M is a Frattini extension and therefore, by 3.3, so is $\varphi_M\gamma$. If $(h, c) \in \ker \varphi_M\gamma$ then $h\gamma = 1$ so $c\delta = h\gamma\alpha = 1$: thus $\ker \varphi_M\gamma$ lies in the direct product of A and ker δ and hence ker $\varphi_M \gamma \in \mathfrak{B}$. The $\mathfrak{B}$-maximality of γ now implies that φ_M is an isomorphism. Put $\beta = \varphi_M^{-1} \psi_M$: as $\varphi\gamma\alpha = \psi\delta$, we have that $\beta\delta = \gamma\alpha$, and then the surjectivity of β follows by 3.2. ■

4. The profinite case. In this section we consider only profinite groups and continuous homomorphisms; the reader is reminded that Frattini subgroups are now taken to be the intersections of the maximal open subgroups.

4.1. *If φ: $F \to G$ is a continuous, surjective homomorphism of profinite groups, then there exists a closed subgroup P in F such that the restriction φ_P: $P \to G$ is a Frattini extension.*

In view of the profinite analogue of 3.1 (for its proof, use Corollary 1 of Lemma 3 in Gruenberg [4], noting that all subgroups considered must be closed now), it is sufficient to establish that the set $\mathscr{S}$ of those closed subgroups P of F which satisfy $P\varphi = G$, has a minimal element. Zorn's Lemma gives this, provided $\bigcap \mathscr{C} \in \mathscr{S}$ for every chain $\mathscr{C}$ in $\mathscr{S}$. Suppose $\mathscr{C}$ is a chain in $\mathscr{S}$ with $\bigcap \mathscr{C} \notin \mathscr{S}$: then for some element g of G the complete inverse image $g\varphi^{-1}$ of g avoids $\bigcap \mathscr{C}$. As G is compact, it follows that $g\varphi^{-1} \cap (\bigcap \mathscr{C}_0) = \emptyset$ for some finite subset $\mathscr{C}_0$ of $\mathscr{C}$. However, as $\mathscr{C}$ is a chain, we have $\bigcap \mathscr{C}_0 \in \mathscr{C} \subseteq \mathscr{S}$, so $g \in (\bigcap \mathscr{C}_0)\varphi$. This contradiction completes the proof of 4.1.

Proof of 2.4. Let G be any profinite group. An unrestricted free profinite group of sufficiently large rank has a continuous homomorphism φ onto G. By 4.1, the restriction of φ to a suitable closed subgroup P of F is a Frattini extension. The projectivity of P will follow from 2.5. ■

Proof of 2.5. By Theorem 4 of Gruenberg [4], projectivity is equivalent to having cohomological dimension at most 1. The profinite version of the Shapiro Lemma (see Proposition 10 on page I-12 of Serre [8]) shows that the cohomological dimension of a closed subgroup cannot exceed that of the whole group. ■

5. Another proof of 2.5. Some readers may like to see how 2.5 may be derived without cohomology. It seems that the role of cohomology in the proof above is merely to facilitate the use of induced modules. For, we show here that the construction of wreath products, which lies behind induced modules in any case, can also be exploited directly.

To start with, we need to recall some familiar facts in the context of abstract (rather than profinite) groups. If S is a permutation group on a set Σ and A is any group, the (unrestricted, permutational) wreath product $A \operatorname{Wr} S$ is constructed as the split extension by S of the group A^{Σ} of all maps from Σ to A, with the usual action. If $B \leqq A$ and $R \leqq S$, then $B \operatorname{Wr} R$ will be regarded a subgroup of $A \operatorname{Wr} S$. Note that the "diagonal" embedding δ: $A \to A^{\Sigma}$ (where $a\delta$ maps all of Σ to the singleton $\{a\}$) has the property that its image $A\delta$ avoids, and lies in the centralizer of, S in $A \operatorname{Wr} S$, so the direct product $(A\delta) \times S$ is a subgroup of $A \operatorname{Wr} S$. To each homomorphism α: $A \to B$ there is a homomorphism $\alpha \operatorname{Wr} S$: $A \operatorname{Wr} S \to B \operatorname{Wr} S$, which is surjective whenever α is surjective (cf. 22.11 in Hanna Neumann's [6]). A less familiar but equally obvious observation is that if S has a fixed point in Σ and one denotes by S_1 the restriction of S to the set obtained from Σ by deleting that fixed point, then each $A \operatorname{Wr} S$ has a natural direct decomposition as $A \times (A \operatorname{Wr} S_1)$. More specifically, we shall make use of the corresponding projections $\pi_A\colon A \operatorname{Wr} S \to A$ (with kernel $A \operatorname{Wr} S_1$), which are such that $\pi_A \alpha = (\alpha \operatorname{Wr} S)\pi_B$ for every $\alpha\colon A \to B$.

The central result about wreath products is the Embedding Theorem, which goes back to the construction of monomial and induced representations by Frobenius. If $H \leqq G$, choose Σ as the set of right cosets of H in G, and denote by σ the obvious homomorphism from G into the symmetric group on Σ. The theorem asserts that there exist embeddings $\varphi\colon G \to H \operatorname{Wr} G\sigma$. Perhaps the most convenient way to describe φ is the following. Clearly, $g \mapsto (g\delta)(g\sigma)$ defines an embedding of G into the direct product $(G\delta) \times (G\sigma)$ and hence also into $G \operatorname{Wr} G\sigma$ which, as noted above, contains this direct product. Choose a set of representatives of the right cosets of H in G: this may be regarded a map from Σ to G (taking each coset to its representative), and hence as an element t of $G \operatorname{Wr} G\sigma$. Follow the embedding $G \to G \operatorname{Wr} G\sigma$, $g \mapsto (g\delta)(g\sigma)$ by the inverse of the inner automorphism of $G \operatorname{Wr} G\sigma$ induced by t. Check that the image of G under the composite map lies in the subgroup $H \operatorname{Wr} G\sigma$ of $G \operatorname{Wr} G\sigma$ (this is a single line of formal calculation, once a convenient notation is established). One can now obtain φ just by restricting the codomain of this composite map.

Of course, φ depends on the choice of t. For the sequel, choose t so that it represents the trivial coset by a central element of H. It is clear that $H\varphi$ lies in $H \operatorname{Wr} H\sigma$. The permutation group $H\sigma$ leaves the trivial coset fixed, so we have the corresponding projection $\pi_H\colon H \operatorname{Wr} H\sigma \to H$ at our disposal. The point of choosing a central representative for the trivial coset is to ensure that the restriction $\varphi_*\colon H \to H \operatorname{Wr} H\sigma$ of φ followed by π_H is the identity map on H (this may also be verified in a single line).

The calculations we suppressed are irrelevant for checking that if Σ is finite and all our data come from the category of profinite groups and continuous homomorphisms (note this means that for the Embedding Theorem H must be an open subgroup of G), all this can be carried out within that category. That done, one can readily see that if G is projective so is H. Indeed, take $\alpha\colon A \to B$ and $\gamma\colon H \to B$ in that category, with α surjective. The projectivity of G ensures the existence of a $\psi\colon G \to A \operatorname{Wr} G\sigma$ with $\psi(\alpha \operatorname{Wr} G\sigma) = \varphi(\gamma \operatorname{Wr} G\sigma)$. For the restrictions $\varphi_*\colon H \to H \operatorname{Wr} H\sigma$ and $\psi_*\colon H \to A \operatorname{Wr} H\sigma$, this yields $\psi_*(\alpha \operatorname{Wr} H\sigma) = \varphi_*(\gamma \operatorname{Wr} H\sigma)$. Now use the projections π_A, π_B, π_H associated with the obvious fixed point of the permutation group $H\sigma$, to obtain that

$$\psi_* \pi_A \alpha = \psi_*(\alpha \operatorname{Wr} H\sigma)\pi_B = \varphi_*(\gamma \operatorname{Wr} H\sigma)\pi_B = \varphi_* \pi_H \gamma = \gamma .$$

Thus $\chi\colon H \to A$ defined as $\psi_* \pi_A$ is such that $\chi\alpha = \gamma$.

Suppose now that P is any closed subgroup of a projective profinite group G. To prove P projective, we have to show that to each $\alpha\colon A \to B$ and $\pi\colon P \to B$, with α surjective, there is a $\xi\colon P \to A$ such that $\xi\alpha = \pi$. By Proposition 1 of Gruenberg [4], we may assume that B is finite. Then $\ker \pi$ is open in P, so $M \cap P \leqq \ker \pi$ for some open subgroup M of G. Write N for the intersection of the conjugates of M in G, and put $NP = H$. As $N \cap P \leqq \ker \pi$, there is a $\gamma\colon H \to B$ (with kernel $N \ker \pi$) whose restriction to P is π. Since H is open, by the argument above there exists a $\chi\colon H \to A$ with $\chi\alpha = \gamma$. The restriction of χ will do as the required ξ. This completes the direct proof of 2.5. ■

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