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The Thirty-nine Varieties

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The study of varieties of groups was initiated by the Cambridge doctoral thesis of B. H. Neumann in 1935, and many of the principal contributions to the subject were subsequently made by him and by his mathematical family. It is a joy to greet him on his seventieth birthday, with much love and gratitude, and with a discussion of some results and some open problems on varieties.

After a brief summary in this paragraph, I shall start from scratch and introduce technicalities as slowly as possible, to make at least part of the paper accessible to non-specialists. This is a survey of what we do know and what we would like to know about torsionfree nilpotent varieties. In particular, it will be proved that there are precisely 39 such varieties of class at most five (there are infinitely many of class six), and each of these will be identified. In the course of the proof it will be seen that if a torsionfree locally nilpotent variety does not contain all, and does not consist of, nilpotent groups of class at most four, then it is metabelian. Not only the metabelian, but all the centre-by-metabelian locally nilpotent torsionfree varieties will be completely classified. In addition, for each positive integer c, we know (although in a less specific sense) all torsionfree nilpotent varieties of class c which contain all nilpotent groups of class less than c. Most of the open problems concern torsionfree nilpotent varieties which have only finitely many torsionfree subvarieties. These have many other attractive features, and it seems conceivable that for sufficiently large c there are only finitely many join-irreducible varieties of this kind with class precisely c: only for $c = 6$ are there known to be infinitely many of them.

1. Introduction

A *variety* is a class of groups defined by a set of *laws* (or *identical relators*): if L is a subset of a free group X (of countably infinite rank), the variety defined by L is the class of all groups G with the property that L is in the kernel of every homomorphism of X into G. The set of *all* laws of this variety is the intersection of the kernels of the homomorphisms of X into groups in the variety. This intersection is the subgroup of X generated by the union of the images of L under the endomorphisms of X: the *fully invariant closure* of L in X. Thus varieties are in a one-to-one correspondence with the fully invariant subgroups of X.

For example, if x and y are elements of a free generating set of X, the variety defined by the single law $[x, y]$ $(= x^{-1}y^{-1}xy)$ is the class $\mathfrak{A}$ of all abelian groups, and the set of all laws of $\mathfrak{A}$ is the commutator subgroups X' of X.

Two subsets of X are called *equivalent* if they define the same variety: that is, if they have the same fully invariant closure. One of the striking results in B. H. Neumann's thesis [21], [22] was that each nontrivial element of X is equivalent to a two-element subset consisting of a power of x and an element of X'. It follows readily that each subset of X is equivalent to a subset of X' augmented by a single power of x. This in turn implies that every variety which consists of abelian groups is definable by the commutative law $[x, y]$ and an exponent law x^n: to each nonnegative integer n, there is the variety $\mathfrak{A}_n$ which consists of the abelian groups in which the orders of elements all divide n, and there are no other abelian varieties. (It is customary to write $\mathfrak{A}_0$ simply as $\mathfrak{A}$, and $\mathfrak{A}_1$ as $\mathfrak{E}$; the latter is just the class of all groups of order 1.) Another immediate consequence is that every variety which does not contain $\mathfrak{A}$ must have finite exponent.

The best-known problem in variety theory has been the *finite basis problem*, first raised in B. H. Neumann's thesis (though its scope had grown in the course of time): can each variety be defined by a finite set of laws? It took until 1969 before the answer was proved to be negative, by Ol'šanskiĭ [26]. It follows (from the details of that answer) that the cardinality of the set of all varieties is that of the continuum: a positive answer would have implied countability. There is a wealth of literature on this problem, giving positive answers for various restricted classes of varieties, and by now numerous examples to show that the positive answers cannot be extended much further.

The abelian case of the problem was, of course, dealt with by B. H. Neumann's result, as discussed above. The next step was the positive answer for the nilpotent case (Lyndon [19]), which is best handled, as in Hanna Neumann's book [24], by starting with a generalization of this result. This generalization is that each subset of X is equivalent to a subset of $X_c \cup \mathfrak{N}_c(X)$ where X_c is the subgroup of X generated by a c-element subset of a free generating set of X while $\mathfrak{N}_c(X)$ is the $(c + 1)$th term of the lower central series of X, that is, the set of all laws of the variety $\mathfrak{N}_c$ of all nilpotent groups of class at most c. For $c = 1$ this is just the result discussed earlier. However, there is a vast difference between the cyclic X_1 and the X_c with $c > 1$, so the reduction in the general case is very much less effective. While it is good enough to yield the finite basis result, it is only a small step towards a classification of nilpotent varieties which would match the complete and decisive nature of the abelian results.

Let us look at this a little closer. The $c = 1$ case tells us that we only need to consider exponent laws and commutator laws. Apply next the case $c = 2$ to the latter: commutator laws are equivalent to two-variable commutator laws (that is, elements of X_2') and commutator laws of higher weight (elements of $\mathfrak{N}_2(X)$). One can now exploit that X_2' is cyclic modulo $\mathfrak{N}_2(X)$, and complete the classification of nilpotent varieties of class 2 without much trouble. In trying for class 3, our luck runs out, for $\mathfrak{N}_2(X_3)$ is not cyclic modulo $\mathfrak{N}_3(X)$. Still, the difficulties are manageable, and varieties of class 3 have also been classified (Jónsson [12], Remeslennikov [27]). The class 4 case is vastly more complicated; it is only now being sorted out by P. Fitzpatrick.

The general case would require us, first, to deal with $\mathfrak{N}_{c-1}(X_c)$ modulo $\mathfrak{N}_c(X)$, and then, harder still, to fuse the conclusions with the results obtained for class $c - 1$. An equivalent form of the first question is to ask for the fully invariant subgroups of X_c sandwiched between $\mathfrak{N}_{c-1}(X_c)$ and $\mathfrak{N}_c(X_c)$. There is, in fact, a deep and largely classical theory applicable to certain aspects of this, which I shall refer to as 'small class theory'. The key ideas were developed in the late thirties and early forties, then consolidated by Graham Higman [11] in 1965; for a more detailed and up-to-date exposition, see a very recent paper of mine [15]. The quotient $\mathfrak{N}_{c-1}(X_c)/\mathfrak{N}_c(X_c)$ is a free abelian group of finite rank which is a module with respect to the semigroup of endomorphisms of X_c, and the fully invariant subgroups we are interested in correspond to the submodules of this module. The theory deals conclusively with the submodules of prime-power index p^k as long as $c < p$ (hence the term 'small class'), and also with the isolated submodules (that is, those which give torsionfree factormodules). However, in general it is virtually unable to touch, say, the submodules of 2-power index.

This is largely in accord with the state of the art in general group theory (as distinct from variety theory). Finitely generated torsionfree nilpotent groups are just about the best understood infinite nonabelian groups, and finite p-groups of class less than p (for any prime p) are similarly well behaved. By comparison, our understanding of finite p-groups without a class bound is extremely meagre (even if we restrict attention, say, to the metabelian case).

The seventies have seen a marked decline of activity in variety theory, and this is frequently blamed on the solution of the finite basis problem. True enough, that was *the* problem which caught the attention of algebraists, and with its solution the glittering bait is gone. It was also the last of what I regard the three fundamental problems of variety theory. (The first was the structure of the multiplicative semigroup of all varieties: shown to be free with zero and identity, by three Neumanns [23] and Šmel'kin [29]; the second, the question of the distributivity of the lattice of all varieties, settled negatively by Higman [11].) Still, I feel that this decline also has a lot to do with variety theory having largely caught up with the rest of group theory, in the sense illustrated by the previous paragraph. While there were plenty of basic unanswered questions concerning varieties of otherwise well understood groups, the subject could move forward with confidence. Today one is impressed by tantalizing varietal questions more or less tied up with well-known unsolved problems of general group theory. For instance, there is the question whether there exist nonabelian varieties in which all finite groups are abelian. Intuitively this appears closely related to the existence problem of Tarski monsters: infinite groups in which every proper nontrivial subgroup has prime order. After the very recent announcement that Rips has made some Tarski monsters, one eagerly awaits details to see whether this will open the way to 'pseudo-abelian' varieties. I also wonder whether it brings us any closer to solving another, so far entirely intractable problem: if a finite group G is contained in the join $\mathfrak{U} \vee \mathfrak{B}$ of two varieties, must G lie in a variety generated by one finite group from $\mathfrak{U}$ and one from $\mathfrak{B}$? (The first of these questions is on record in Hanna Neumann's book [24]; the second has also been around for a long time, though its only occurrence in print that I know of is in [14].)

One of the aims of this paper is to draw attention to a number of apparently unexplored questions which do not carry the discouraging label 'well known to be hard'. To motivate them and to make them more accessible, I need to spell out some of the rich folklore of the subject. The results I present were broadly recognized a decade ago, certainly in my discussions with M. F. Newman (to whom I am greatly indebted) and very likely by many others, yet as far as I know most of them have never been actually formulated, let alone printed. It is not my intention to claim credit or priority, and I regret if I have overlooked some relevant publication and so fail to give acknowledgement where it is due.

2. Torsionfree varieties

In calling a variety *torsionfree* one does not use the adjective in the same sense as in the case of abelian or nilpotent varieties, for of course only the trivial variety $\mathfrak{E}$ consists of torsionfree groups. What is meant here is a variety which is generated by torsionfree groups, or (equivalently) whose free groups are torsionfree. Thus the torsionfree varieties are precisely those which correspond to isolated fully invariant subgroups of the free group X. This terminology is a convenient start towards exploring what advantage may be taken, in the context of varieties, of the general fact that torsionfree groups are better behaved than arbitrary groups.

Our descent into technicalities must accelerate now: the reader will soon need most of the basic language of varieties, still much the same as in Hanna Neumann's monograph [24]. The bulk of this section will be taken up by examining, in broad outline, how the structure of the set (call it T) of all torsionfree varieties compares with the structure of the set (call it Γ) of all varieties of groups.

First recall that Γ is a free semigroup with zero $\mathfrak{O}$ (the variety of all groups) and identity $\mathfrak{E}$. Obviously, $\mathfrak{O}$ and $\mathfrak{E}$ lie in T, and 22.22 of [24] gives that T is a subsemigroup of Γ. In fact, much more is true: *a product of two varieties (other than $\mathfrak{E}$ and $\mathfrak{O}$) is torsionfree if and only if the first factor is torsionfree.* Before proving this. let us note that not only does this mean that T (or rather, T without $\mathfrak{E}$) is a right ideal in Γ, but also that T itself is a free semigroup with zero and identity. (The deduction of the latter is straightforward: if $\mathfrak{O} \neq \mathfrak{U} \in \mathrm{T}$ and $\mathfrak{U}_1\mathfrak{U}_2\cdots\mathfrak{U}_n$ is the unique factorization of $\mathfrak{U}$ with each factor indecomposable in Γ, the unique factorization of $\mathfrak{U}$ with each factor indecomposable in T is $(\mathfrak{U}_{t(1)}\cdots\mathfrak{U}_{t(2)-1})(\mathfrak{U}_{t(2)}\cdots\mathfrak{U}_{t(3)-1})\cdots(\mathfrak{U}_{t(k)}\cdots\mathfrak{U}_n)$ where $\mathfrak{U}_{t(i)}$ is the ith among the torsionfree members of the sequence $\mathfrak{U}_1, \mathfrak{U}_2, \ldots, \mathfrak{U}_n$.) Moreover, $\mathfrak{U} \mapsto \mathfrak{A}\mathfrak{U}$ defines a map of Γ into T: as this map is one-to-one (on account of the uniqueness of factorization in Γ), it follows that the cardinality of T is also that of the continuum. Thus T and Γ are isomorphic as semigroups.

It remains to prove the italicized statement of the previous paragraph. The 'only if' part is immediate from the fact that the free groups of $\mathfrak{U}$ are subgroups of the free groups of $\mathfrak{U}\mathfrak{V}$. For the proof of 'if', let $\mathfrak{E} \neq \mathfrak{U} \in \mathrm{T}$ and $\mathfrak{O} \neq \mathfrak{V} \in \Gamma$; let V be the verbal subgroup of X corresponding to $\mathfrak{V}$, and $\mathfrak{U}(V)$ the verbal subgroup of the free group V corresponding to $\mathfrak{U}$: we have to show that $X/\mathfrak{U}(V)$ is torsionfree. By assumption, $\mathfrak{A} \leq \mathfrak{U}$ so $\mathfrak{U}(V) \leq V'$, and $V/\mathfrak{U}(V)$ is torsionfree. Thus $V'/\mathfrak{U}(V)$ is torsionfree and so if X/V' is torsionfree we are done. However, it is an elementary fact that X/V' is torsionfree for every normal subgroup V of every free group X. (As I cannot think of a reference, here goes the proof. If w is an element of X with $w^k \in V'$ for some positive integer k, consider the subgroup W of X generated by V and w: this is free, so W/W' is torsionfree, and thus $w^k \in V' \leq W'$ implies that $w \in W'$; but $W' \leq V$ and V/V' is torsionfree, so $w \in V'$ follows.)

Next, recall that Γ is a complete modular lattice with respect to partial order by inclusion. Of course T inherits this partial order from Γ, and as the intersection of any set of isolated fully invariant subgroups is isolated and fully invariant, T is also a complete lattice with respect to this partial order. However, while joins in T are the same as joins in Γ (so T is a sub-join-semilattice of Γ), the meet in T of two elements of T is at times smaller than the set-theoretic intersection which is their 'meet' in Γ. This reflects the fact that not all products of isolated verbal subgroups of X are isolated. It follows that T is *not* a sublattice of Γ. (*In the rest of this paper, I shall consistently use 'meet' and* $\wedge$ *in* T *but 'intersection' and* $\cap$ *in* Γ, *departing in this respect from standard usage.*) The question cries out for answer: *is* T *modular*? I have made no serious attempt to answer this in general, and leave it as our FIRST OPEN PROBLEM. The modularity of the sublattice of T consisting of the torsionfree nilpotent-by-abelian varieties will be more than enough for the present, and that much is easily established. (As every torsionfree variety other than $\mathfrak{E}$ contains $\mathfrak{A}$, the corresponding verbal subgroups of X may as well be assumed to lie inside X', and modulo any one of them X' is nilpotent. Then one exploits the fact that in a nilpotent group the elements of finite order form a subgroup, so the 'isolator' of the product of two of these subgroups has a particularly simple description.) Another property of Γ (uses of this tend to be implicit in the literature) is that it is *join-continuous*, that is, $(\bigcap_\lambda \mathfrak{U}_\lambda) \vee \mathfrak{V} = \bigcap_\lambda (\mathfrak{U}_\lambda \vee \mathfrak{V})$ whenever the $\mathfrak{U}_\lambda$ form a chain. (Unfortunately, many authors use 'join-continuity' to mean the dual property.) From the dual version of this law for fully invariant subgroups of X, one readily sees that T is also join-continuous.

For any variety $\mathfrak{V}$, let $\Lambda(\mathfrak{V})$ denote the lattice of all subvarieties of $\mathfrak{V}$, and $\Lambda^0(\mathfrak{V})$ the lattice of all torsionfree subvarieties of $\mathfrak{V}$. (Thus $\Lambda(\mathfrak{V})$ is a sublattice of Γ, and $\Lambda^0(\mathfrak{V})$ is a sublattice of T: it will be convenient to have the latter defined even when $\mathfrak{V}$ itself is not torsionfree.) It is known that T *is not distributive*, for small class theory tells us that $\Lambda^0(\mathfrak{N}_6)$ is not distributive. In fact, even just the qualitative part of the 'Torsionfree Classification Theorem' in [15] yields something of interest here: *for nilpotent* $\mathfrak{V}$, *we know that* $\Lambda^0(\mathfrak{V})$ *is distributive if and only if it is finite.* The derivation consists of two simple observations. Let $\mathfrak{V}$ be a torsionfree subvariety of $\mathfrak{N}_c$, say; then $\Lambda^0(\mathfrak{V})$ is the interval bounded by $\mathfrak{E}$ and $\mathfrak{V}$ in $\Lambda^0(\mathfrak{N}_c)$, that is $\Lambda^0(\mathfrak{V}) = \{\mathfrak{U} \in \Lambda^0(\mathfrak{N}_c) \mid \mathfrak{E} \leq \mathfrak{U} \leq \mathfrak{V}\}$. The first observation is that for any lattice homomorphism, the image of an interval is an interval in the image. Now the Torsionfree Classification Theorem states that $\Lambda^0(\mathfrak{N}_c)$ is the subdirect product of the subspace lattices of finitely many rational vector spaces: so all that remains is to observe that an interval in such a subspace lattice is itself the subspace lattice of some space and is therefore distributive if and only if it is finite.

As to the *finite basis problem* for torsionfree varieties, we already have the negative answer: we have seen that there are uncountably many torsionfree varieties, while of course X has only countably many finite subsets. Of course, this does not mean that there are no questions left, but I leave it to the reader to examine which of the many positive or negative results in this direction have easy or challenging torsionfree analogues. (A survey of most of the relevant literature can be found in [16].) Let us call this group of questions our SECOND OPEN PROBLEM.

A notion which may be relevant here (and which will certainly ease our burden in later sections of this paper) is the following. In any group, the *isolator* of a normal subgroup is the intersection of the isolated normal subgroups which contain it. Another description of the isolator is the following. For a normal subgroup N of a group G, define N^* by saying that N^*/N is the subgroup generated by all the elements of finite order in G/N; put $N_1 = N$, define inductively $N_{i+1} = N_i^*$, and let $N_\omega = \bigcup_i N_i$: then N_ω is the isolator of N in G. From this it is easy to see that if N is fully invariant in G, so is N_ω and each of the N_i. Now a torsionfree variety *as such* may be said to be defined by a subset L of X if the corresponding verbal subgroup of X is the isolator of the fully invariant closure of L. It is possible that some torsionfree variety may be definable by a finite L in this sense, without having a 'finite basis' in the usual sense: indeed, if the infinite ascending chain of fully invariant subgroups (in the second description of the isolator of the fully invariant closure of some finite L) is properly ascending all the way, then that must be the case. (I have not looked for examples.) Thus one might expect stronger positive finite basis results in this sense. (Of course, the cardinality considerations prove that the general answer remains negative.)

The term corresponding to *isolator* in the context of varieties is *torsionfree core:* for any variety $\mathfrak{B}$, this is the subvariety $\mathfrak{B}_0$ generated by all the torsionfree groups in $\mathfrak{B}$. Thus for any subset L of X, the torsionfree variety defined as such by L is the torsionfree core of the variety defined by L.

I shall not continue to explore any further what may correspond in T to the 'algebra of varieties' Γ in the sense of Section 1 of Chapter 2 in Hanna Neumann's [24]: for some of the results recorded there, it does not seem obvious to decide what the torsionfree analogues might be. There are, however, two deep results which must be borne in mind. One is that T is *not* closed with respect to commutator formation, for C. K. Gupta has shown [7] that the variety $[\mathfrak{A}^2, \mathfrak{E}]$ of all centre-by-metabelian groups is not torsionfree. The other is that cutting back to the torsionfree core of a commutator is a delicate operation. While the torsionfree core of $[\mathfrak{B}_{2^n}, \mathfrak{E}]$ is easily seen to be just $\mathfrak{A}$ (Narain Gupta and A. Rhemtulla [8]), Adjan showed [1] that for every large prime p the torsionfree core of $[\mathfrak{B}_p, \mathfrak{E}]$ contains $\mathfrak{B}_p$ as well.

What we have found supports the reader who reacted with scepticism to my unqualified claim that torsionfree groups are better behaved than groups in general: while that is so in the nilpotent or metabelian case, without some such restriction torsionfree groups seem to form no less complicated a picture than the general one. This is still a wildly intuitive statement, but such thoughts do form part of our overall view of group theory. I hope this section illustrates that varieties offer, even to those who are not interested in varieties for their own sake, a language in which such matters may be explored on a less intuitive level.

3. Counting to thirty-nine

It is time to turn from generalities to specifics: just what individual torsionfree varieties do we know? As observed in the previous section, a product variety is torsionfree if (and only if) its first factor is, so the real task is to look at indecomposable torsionfree varieties (that is, those which have no proper factorization even in Γ). All nilpotent varieties are indecomposable, so they provide a good starting point (not that this is the only reason for giving them precedence).

One aim of this section is to count up that *there are precisely* 39 *torsionfree nilpotent varieties of class at most* 5. Closer identification of all but nine is deferred to the next section. According to the general program laid down in the introduction, the problem breaks up into two parts. The first is to count, for $c \leq 5$, the isolated fully invariant subgroups of X_c which lie between $\mathfrak{N}_{c-1}(X_c)$ and $\mathfrak{N}_c(X_c)$. The Torsionfree Classification Theorem of small class theory [15] tells us that the lattice of these is the direct product of the subspace lattices of certain rational vector spaces. For $c \leq 5$ each of these spaces is 1-dimensional so each subspace lattice has just two elements. The number of direct factors is 1 when $c \leq 3$, but 2 for $c = 4$ and 5 for $c = 5$. Thus the number of torsionfree varieties strictly between $\mathfrak{N}_{c-1}$ and $\mathfrak{N}_c$ is 0 for $c \leq 3$, it is 2 $(= 2^2 - 2)$ for $c = 4$, and 30 $(= 2^5 - 2)$ for $c = 5$. Of course each $\mathfrak{N}_c$ is torsionfree, so with $\mathfrak{E}, \mathfrak{A}, \mathfrak{N}_2, \mathfrak{N}_3, \mathfrak{N}_4, \mathfrak{N}_5$ the count is already up to 38.

The second step of the program is to complete the picture by counting the torsionfree subvarieties of $\mathfrak{N}_5$ which do not lie in any of the intervals already investigated. Small class theory is not much help in this beyond telling us that $\Lambda^0(\mathfrak{N}_5)$ is distributive and finite (and providing a very crude upper bound for its number of elements), so we have to fall back on *ad hoc* arguments. These are easy enough to find here, but do not seem to be so readily available should one attempt, say, the class 6 case. In fact, I do not even know *how many torsionfree subvarieties of* $\mathfrak{N}_6$ *fail to contain* $\mathfrak{N}_5$: let this be our THIRD OPEN PROBLEM. It is perfectly feasible that this number could be finite, but our ignorance here is in stark contrast with what we are about to prove for the class 5 case. (One way to approach this problem would be to concentrate on the torsionfree subvarieties of $\mathfrak{N}_6$ which are maximal with respect to not containing $\mathfrak{N}_5$. It is not hard to see that there are precisely five such varieties, one for each maximal subvariety of $\mathfrak{N}_5$: just use the distributivity, proved in Section 2 of [15], of the sublattice of $\Lambda^0(\mathfrak{N}_6)$ generated by $\mathfrak{N}_5$ and any two other elements. If these five, or rather, their joins with $\mathfrak{N}_5$, could be identified in the setting of the Torsionfree Classification Theorem, the answer to the problem could be read off the information already available. The only thing known in this direction is that one of the five varieties contains the rank 2, but not the rank 3, free group of $\mathfrak{N}_6$: Chau [4], [5].)

To justify the count of 39, we must show that there is just one further torsionfree subvariety in $\mathfrak{N}_5$. It is, in fact, $\mathfrak{N}_5 \cap \mathfrak{A}^2$. (We can write intersection in place of meet, for a result of Magnus, 36.32 in Hanna Neumann's [24], implies that each $\mathfrak{N}_c \cap \mathfrak{A}^2$ is torsionfree.) As a first step towards this, we need to identify the two torsionfree varieties strictly between $\mathfrak{N}_3$ and $\mathfrak{N}_4$. Further details of small class theory could be invoked here, but it is hardly necessary to conduct a systematic search to spot these two old friends. One is, of course, $\mathfrak{N}_4 \cap \mathfrak{A}^2$: note that this contains, and is in fact generated by, the rank 2 free group of $\mathfrak{N}_4$. The other is the torsionfree core of the variety $\mathfrak{N}_3^{(2)}$ defined by the 2-variable laws of $\mathfrak{N}_3$. This can also be defined, *as torsionfree variety*, by the third Engel law $[y,x,x,x]$, so I shall denote it by $\mathfrak{E}_3$. Indeed, Heineken proved [9] that torsionfree third Engel groups have class at most 4, and for instance the simplest case of the general construction at the end of Newman's [25] shows the existence of such groups of class precisely 4: since we know that there are precisely two torsionfree varieties strictly between $\mathfrak{N}_3$ and $\mathfrak{N}_4$, this is sufficient to establish the claim.

The other steps can be made, at very little extra cost, much more general than what is needed just for counting to 39. It will be more convenient to break pattern and display them.

Lemma 1. *If* $\mathfrak{B}$ *is any torsionfree locally nilpotent variety such that* $\mathfrak{B} \wedge \mathfrak{N}_{c+1} \leq \mathfrak{N}_c^{(r)}$, *then* $\mathfrak{B} \leq \mathfrak{N}_c^{(r)}$.

That is, if every r-generator torsionfree group of class at most $c + 1$ in $\mathfrak{B}$ in fact has class at most c, then every r-generator group in $\mathfrak{B}$ has class at most c.

To see this, let F be the rank r free group of $\mathfrak{B}$; by assumption, this is torsionfree and nilpotent, say of class precisely d. If $d \leq c$ we are done; so suppose $d > c$ and aim for a contradiction. Let Z_{d-c-1} be the $(d-c-1)$th term of the upper central series of F. As all terms of the upper central series of torsionfree nilpotent groups are isolated (cf. 31.41 in [24]), F/Z_{d-c-1} is an r-generator torsionfree group of class precisely $c+1$ in $\mathfrak{B}$, contrary to our hypothesis.

Lemma 2. *If* $\mathfrak{B}$ *is a torsionfree locally nilpotent variety such that* $\mathfrak{B} \wedge \mathfrak{N}_4 \leq \mathfrak{A}^2$, *then* $\mathfrak{B} \leq \mathfrak{A}^2$.

The proof of this is somewhat longer. Before imposing it on the reader, let me show that it is worth the trouble, for it leads quickly to a result that does a lot more than complete the count.

Theorem. The only torsionfree locally nilpotent varieties which do not contain $\mathfrak{N}_4$ *are* $\mathfrak{C}_3$ *and the* $\mathfrak{N}_c \cap \mathfrak{A}^2$.

(Of course, for $c < 4$ the latter are just $\mathfrak{E}, \mathfrak{A}, \mathfrak{N}_2$, and $\mathfrak{N}_3$.)

The only additional information we need for the proof of this is the fact that *the metabelian torsionfree varieties are precisely the* $\mathfrak{N}_c \cap \mathfrak{A}^2$. In fact, M. F. Newman and I have classified all torsionfree metabelian varieties (see Bryce [3]): apart from $\mathfrak{A}^2$ itself, they are all possible finite joins of the varieties of the form $\mathfrak{N}_c\mathfrak{A}_n \cap \mathfrak{A}^2$, and each of them has a unique irredundant expression as such a join. However, that result is far too heavy a weapon to use here; what we need can be had much more directly, for instance as follows. Let $\mathfrak{U}$ be any metabelian torsionfree nilpotent variety: say, $\mathfrak{U} \leq \mathfrak{N}_c \cap \mathfrak{A}^2$. Then $\mathfrak{U}$ is generated by its free group F of rank c, and F is residually of prime exponent p with p ranging through any infinite set Π of primes (Higman [10]), so $\mathfrak{U} = \vee \{\mathfrak{B}_p \cap \mathfrak{U} \mid p \in \Pi\}$. It is well known that every metabelian nilpotent variety of prime exponent p and class less than p is of the form $\mathfrak{B}_p \cap \mathfrak{N}_{c'} \cap \mathfrak{A}^2$ (Brisley [2] and Weichsel [31]; also 54.27 in [24]). Thus $\mathfrak{B}_p \cap \mathfrak{U} = \mathfrak{B}_p \cap \mathfrak{N}_{c'(p)} \cap \mathfrak{A}^2$ when $p > c$. As the function c' has finite range, it must be constant, say with value d, on some infinite set Π of primes. Higman's result applies equally well to the rank d free group of $\mathfrak{N}_d \cap \mathfrak{A}^2$ in place of F, so we may conclude that

Should be: the metabelian nilpotent torsionfree ...

$$\mathfrak{U} = \vee \{\mathfrak{B}_p \cap \mathfrak{U} \mid p \in \Pi\} = \vee \{\mathfrak{B}_p \cap \mathfrak{N}_d \cap \mathfrak{A}^2 \mid p \in \Pi\} = \mathfrak{N}_d \cap \mathfrak{A}^2.$$

For the proof of the theorem, let $\mathfrak{B}$ be a torsionfree locally nilpotent variety not containing $\mathfrak{N}_4$, and let G be the rank 2 free group of $\mathfrak{B} \wedge \mathfrak{N}_4$. All 2-generator groups of class at most 4 are metabelian, so the variety generated by G is $\mathfrak{N}_d \cap \mathfrak{A}^2$ for some d with $d \leq 4$. If $d \leq 2$, we have that $\mathfrak{B} \wedge \mathfrak{N}_4 \leq \mathfrak{N}_2^{(2)}$: this is a rather trivial case, as the torsionfree core of $\mathfrak{N}_2^{(2)}$ is just $\mathfrak{N}_2$ (Levi; see 34.31 in Hanna Neumann's [24]), and $\mathfrak{B} = \mathfrak{N}_d \cap \mathfrak{A}^2$ follows by an even simpler version of the proof of Lemma 1. If $d \geq 3$, we first note that $\mathfrak{N}_3 \leq \mathfrak{N}_d \cap \mathfrak{A}^2 \leq \mathfrak{B} \wedge \mathfrak{N}_4 < \mathfrak{N}_4$. If $d = 3$, we use Lemma 1 with the conclusion that $\mathfrak{B}$ is either $\mathfrak{N}_3$ $(= \mathfrak{N}_3 \cap \mathfrak{A}^2)$ or $\mathfrak{C}_3$ (the torsionfree core of $\mathfrak{N}_3^{(2)}$). If $d = 4$ we have that $\mathfrak{B} \wedge \mathfrak{N}_4 = \mathfrak{N}_4 \cap \mathfrak{A}^2$ so, by Lemma 2, $\mathfrak{B}$ is metabelian. Since $\mathfrak{B}$ is torsionfree locally nilpotent, each of its finite rank free groups generates an $\mathfrak{N}_c \cap \mathfrak{A}^2$, and between them they generate $\mathfrak{B}$. However, the join of any infinite set of $\mathfrak{N}_c \cap \mathfrak{A}^2$ is $\mathfrak{A}^2$ (since free metabelian groups are residually nilpotent: Gruenberg [6]; cf. also 26.32 in [24]) which is not locally nilpotent, so $\mathfrak{B}$ must be some $\mathfrak{N}_c \cap \mathfrak{A}^2$ as claimed.

It might be noted here that this last step, showing that a metabelian torsionfree locally nilpotent variety must in fact be nilpotent, together with a version of a well-known result of P. Hall, yields that a nonnilpotent, locally nilpotent, torsionfree variety would have to be insoluble: the existence of such varieties is a hard problem of long standing (Question 4 in [16]; for a detailed discussion, see Kostrikin [13]).

The proof of Lemma 2 (inspired by Stewart [30]) is of an entirely different nature. So far we have been able to get away with what seems like 'general nonsense' compared to the explicit commutator calculations so common in work on nilpotent varieties, but we must turn to these now. Since the variety of metabelian groups is defined by a 4-variable law, we only need to prove that the rank 4 free group of $\mathfrak{B}$ is metabelian, so no generality is lost by assuming that $\mathfrak{B}$ is actually nilpotent. Induction on its class then reduces our task further: it will be sufficient to prove that if $c \geq 5$ and $\mathfrak{B}$ is a torsionfree subvariety of $\mathfrak{N}_c$ such that $\mathfrak{B} \wedge \mathfrak{N}_{c-1}$ is metabelian, then $\mathfrak{B}$ is metabelian. Towards establishing this, take F to be the rank c free group of $\mathfrak{N}_c$, and let V and N be the (isolated) verbal subgroups of F corresponding to $\mathfrak{B}$ and $\mathfrak{N}_{c-1}$, respectively. We need to show that the second derived group F'' of F is contained in V, given that it is contained in the isolator $\overline{VN}$ of VN. As F/VN is a finitely generated nilpotent group, $\overline{VN}/VN$ is the (finite) subgroup consisting of the elements of finite order in F/VN. Let $\{f_1, \ldots, f_c\}$ be a free generating set of F; by what has been said, some nontrivial power $[f_5, f_4; f_2, f_1]^k$ of the element $[f_5, f_4; f_2, f_1]$ of F'' must lie in VN: thus

$$[f_5, f_4; f_2, f_1]^k = vw \qquad \text{with} \qquad v \in V, w \in N. \tag{$*$}$$

Let γ, δ denote the endomorphisms of F defined by

$$\begin{aligned} f_5\gamma &= [f_5, f_3], & f_i\gamma &= f_i \quad \text{if} \quad i \neq 5, \\ f_5\delta &= 1, & f_i\delta &= f_i \quad \text{if} \quad i \neq 5, \end{aligned}$$

and write $w = w_1 w_2 \cdots w_n$ with each w_j a basic commutator, or the inverse of a basic commutator, of weight c. Apply δ to $(*)$: this gives $(v\delta)(w\delta) = 1$, so $(*)$ may be replaced by

$$[f_5, f_4; f_2, f_1]^k = v(v^{-1}\delta)(w^{-1}\delta)w = v(v^{-1}\delta) \prod_{j=1}^{n} (w_j^{-1}\delta)w_j.$$

Note that $w_j^{-1}\delta$ is 1 or w_j depending on whether f_5 is or is not an entry of w_j, so what is left of $(w^{-1}\delta)w$ after cancellations is the product of the w_j which do involve f_5 as an entry: when next we apply γ, these all increase in weight and hence vanish. It follows that $[f_5, f_3, f_4; f_2, f_1]^k = ((v^{-1}\delta)v)\gamma \in V$. Because V is isolated, this means that $[f_5, f_3, f_4; f_2, f_1] \in V$ so $[\mathfrak{N}_2(F), F'] \leq V$, and it is immediate from this that V must contain all the basic commutators of weight greater than 4 which are not left-normed. In particular, $F'' \cap N \leq V$. It was shown in Section 2 of [15] that the lattice of *isolated* verbal subgroups of F generated by N and any two such subgroups is distributive: thus $F'' \leq \overline{VN}$ gives that $F'' = F'' \cap \overline{VN} = \overline{(F'' \cap V)(F'' \cap N)}$, and so from $F'' \cap N \leq V$ we can conclude that $F'' \leq V$ as required.

This completes the proof of all the claims made in this section, and we are almost ready to move on. In taking stock of the torsionfree nilpotent varieties that are known to us, we have noted those which lie between $\mathfrak{N}_{c-1}$ and $\mathfrak{N}_c$ for some c; those of class at most 5 (isolating the specific problems in the way of dealing with the class 6 case); and the metabelian ones. The latter form part of a larger, also complete picture, namely that of the torsionfree nilpotent centre-by-metabelian varieties. The classification of these runs as follows.

For $k \geq 3$, let $\mathfrak{E}_k$ be defined *as torsionfree variety* by the kth Engel commutator $[y,x,\ldots,x]$ (where x is repeated k times), the nilpotent law of class $k+1$, and the centre-by-metabelian law $[x,y;u,v;w]$. (That is, $\mathfrak{E}_k$ is the torsionfree core of the variety defined by these laws. Note that for $k = 3$ this is consistent with our earlier definition.) The result is that *each torsionfree nilpotent centre-by-metabelian variety is either an* $\mathfrak{N}_c \cap \mathfrak{A}^2$ *or an* $\mathfrak{E}_k$ *or an* $(\mathfrak{N}_c \cap \mathfrak{A}^2) \vee \mathfrak{E}_k$ *with* $k < c$. This expression for a variety is unique, and two such varieties are comparable (in the partial order by inclusion) if and only if that is obvious from writing them in this form. Thus in particular the lattice of all torsionfree nilpotent centre-by-metabelian varieties is distributive. (The torsionfree core of $\mathfrak{N}_c \cap [\mathfrak{A}^2,\mathfrak{E}]$ appears here as $\mathfrak{N}_c \cap \mathfrak{A}^2$ when $c \leq 3$ and as $(\mathfrak{N}_c \cap \mathfrak{A}^2) \vee \mathfrak{E}_{c-1}$ when $c \geq 4$.) These claims are proved either by adapting the whole argument of Stewart [30] from the large prime exponent to the torsionfree case, or by imitating the reasoning we used in the metabelian case to deduce them direct from Stewart's conclusions. (There is one complication in the latter, namely that Stewart used not the torsionfree $\mathfrak{E}_k$ but the ordinary variety defined by the three laws. Fortunately, the torsion part (if any) of the rank $k+1$ free group of that variety must be finite, and it is more than good enough for the applicability of Higman's theorem that, up to any given class, only finitely many primes have to be excluded.)

One might ponder whether this could be taken any further: *is* $\Lambda^0(\mathfrak{N}_c \cap [\mathfrak{A}^2,\mathfrak{E},\mathfrak{E}])$ *also finite, for each* c? Let this be our FOURTH OPEN PROBLEM. If it worked out positively, it would be the last step in this direction, for of course $\mathfrak{N}_6 \leq [\mathfrak{A}^2,\mathfrak{E},\mathfrak{E},\mathfrak{E}]$ and $\Lambda^0(\mathfrak{N}_6)$ is infinite.

4. Counting slowly to five

It is one thing to be able to count that there are precisely 39 torsionfree nilpotent varieties of class at most 5, and quite another to list them individually. Fortunately, those which have not been dealt with so far all lie between $\mathfrak{N}_4$ and $\mathfrak{N}_5$ and so fall into the scope of small class theory. I have been tempted to elaborate here just how that theory copes with this task on its own; for it is quite a complicated theory, hard to assimilate without seeing it at work on specific examples, and I know of no comparable exercise in print except for Stewart's [30]. On balance, it seemed more appropriate to take advantage of what shortcuts offered themselves, though detailed small class theory remains the backbone of the argument.

In the first instance, identification will be by defining sets of laws, the nilpotent law of class 5 being always included even if not specifically mentioned. Small class theory is not capable of giving such definitions except in the sense of defining *torsionfree* varieties: that is, I shall always mean the torsionfree core of the varieties defined by the given laws, even if I omit to repeat this again and again. With some regret, I have to leave as our FIFTH OPEN PROBLEM the removal of this weakness from the results of this section. (The corresponding problem is also open for most of the torsionfree nilpotent centre-by-metabelian varieties discussed before. In the class 5 case, only the $\mathfrak{N}_c$ with $0 \leq c \leq 5$ and $\mathfrak{N}_4 \cap \mathfrak{A}^2, \mathfrak{N}_5 \cap \mathfrak{A}^2$ are adequately covered here in this stronger sense; Chau [4], [5] has dealt with the two varieties generated by the rank 2 and the rank 3 free groups of $\mathfrak{N}_5$; and each of the remaining 29 cases appears to require careful individual consideration.) Other forms of identification may be more revealing in some sense, though they are not available for each of the 39: I shall return to this at the end of the section.

From what has already been said about $\Lambda^0(\mathfrak{N}_5)$, we know that $\mathfrak{N}_5$ has precisely five maximal (proper) torsionfree subvarieties, and the 30 torsionfree varieties strictly between $\mathfrak{N}_4$ and $\mathfrak{N}_5$ are the meets of the 30 proper nonempty subsets of this 5-element set. It is therefore only necessary to give defining sets of laws for the five maximals: in fact, a single defining law will be found for each of the five, and then each of the 30 will be defined (as torsionfree variety) by a proper nonempty subset of the set of these five laws (the nilpotent law of class 5 being always understood as an extra). The discussion concerning our third open problem has already highlighted the importance of the maximal torsionfree subvarieties of $\mathfrak{N}_5$, so it may be convenient also in that context that attention is focussed on them once again.

For many reasons, I would have preferred a different approach: to pick out the join-irreducible subvarieties of $\mathfrak{N}_5$. Each variety we are interested in here is uniquely expressible as an irredundant join of some of these. Join is the same in T as in Γ, and it is easy to see that if a nilpotent variety is join-irreducible in T it is also join-irreducible in Γ. Other classes of varieties have usually been described in terms of join-irreducibles. We already know the centre-by-metabelian join-irreducibles; that leaves only three more to identify, and one of those is the torsionfree core of $\mathfrak{N}_4^{(3)}$. (Of course, if we have finite defining sets of laws for two varieties, we have no way of deriving from these a finite defining set for their join: that would have been a drawback of this approach.) Unfortunately, I have not been able to pin down the other two without going through the whole procedure here: it would be very interesting to have direct identifications for them, in more familiar terms than what can be extracted from the following.

To set the scene for detailed small class theory, a series of further definitions are required. Let A be the algebra of polynomials with rational coefficients in several *noncommuting* variables $x, y, \ldots$: the number of the variables is to be finite but unspecified for the time being. For each nonnegative integer c, let A_c denote the subspace of homogeneous polynomials of (total) degree c in A. The obvious basis of A_c consists of the monomials of degree c. The symmetric group S_c of degree c has a natural action on this basis (permuting monomials by permuting the order of their factors), and hence on the whole of A_c. It is a simple combinatorial exercise to write down the character of S_c afforded by A_c, and then a straightforward application of the orthogonality relations enables one, at least for small values of c, to deduce the structure of A_c as S_c-module. For instance, when A has just 3 variables, one finds that A_5 is the direct sum of 3 irreducibles of type $2^2 1$ (and dimension 5), 6 irreducibles of type 31^2 (and dimension 6), and so on. Here, as usual, the isomorphism types of irreducible S_c-modules are labelled by partitions of c; the symbol $2^2 1$ stands for the partition $2 + 2 + 1$ of 5, and 31^2 for $3 + 1 + 1$. For each partition π of c, let $A(\pi)$ denote the sum of all irreducible S_c-submodules of type π in A_c: then A_c is the direct sum of the $A(\pi)$.

Next, regard A a Lie algebra with respect to the usual bracket product defined by $[a,b] = ab - ba$, and let L be the Lie subalgebra of A generated by r of the variables of A. Put $L_c = L \cap A_c$ and $L(\pi) = L \cap A(\pi)$: then L_c is the direct sum of the $L(\pi)$; and L, of the L_c. For partitions π with more than r parts, $L(\pi) = 0$.

Consider the general linear group $\mathrm{GL}(r, \mathbb{Q})$ acting on L as the group of invertible homogeneous linear substitutions: each $L(\pi)$ is mapped onto itself by these. Torsionfree 'small class' theory is based on the fact ([15], Section 3) that the lattice of $\mathrm{GL}(r,\mathbb{Q})$-submodules of L_c is isomorphic to the lattice of isolated verbal subgroups of X_r between $\mathfrak{N}_{c-1}(X_r)$ and $\mathfrak{N}_c(X_r)$. In particular, the result concerning torsionfree varieties between $\mathfrak{N}_4$ and $\mathfrak{N}_5$, which we have used repeatedly, corresponds to the facts that for $c = 5$ each nonzero $L(\pi)$ is irreducible as $\mathrm{GL}(r, \mathbb{Q})$-module and for $r \geq 4$ precisely five of the $L(\pi)$ are nonzero. It is therefore natural to label the five maximal torsionfree subvarieties of $\mathfrak{N}_5$ by the appropriate partitions so that the verbal subgroup of X_r determined by the variety $\mathfrak{N}(\pi)$ should correspond to $L(\pi)$ in the lattice isomorphism referred to above. (For $c > 5$ there are partitions π with $L(\pi)$ not irreducible: the matching varieties $\mathfrak{N}(\pi)$ may still be of considerable interest, though in that case they are not maximal among the torsionfree subvarieties of $\mathfrak{N}_c$.) Thus the five varieties we seek will be called $\mathfrak{N}(21^3)$, $\mathfrak{N}(2^21)$, $\mathfrak{N}(31^2)$, $\mathfrak{N}(32)$, and $\mathfrak{N}(41)$.

Only one of these five partitions has more than 3 parts: thus the rank 3 free group $F_3(\mathfrak{N}_5)$ of $\mathfrak{N}_5$ lies in $\mathfrak{N}(21^3)$ but in none of the other four varieties. As $\mathfrak{N}_4$ is generated by $F_3(\mathfrak{N}_4)$ which is a factorgroup of $F_3(\mathfrak{N}_5)$, we may conclude that the variety generated by $F_3(\mathfrak{N}_5)$ is precisely $\mathfrak{N}(21^3)$, and so may simply quote a much discussed defining law of this variety from Kovács, Newman and Pentony [17], Levin [18], and Chau [4], [5]: $\mathfrak{N}(21^3)$ *is defined by* (*the class* 5 *law and*) $u(21^3)$ *which may be given as*

$$[z,w,z;y,x][z,x,z;y,w]^{-1}[z,y,z;x,w][y,w,z;z,x]^{-1}[y,x,z;z,w][x,w,z;z,y].$$

The next shortcut comes from an appeal to the results on torsionfree nilpotent centre-by-metabelian varieties near the end of the previous section. Since $(\mathfrak{N}_5 \cap \mathfrak{A}^2) \vee \mathfrak{N}_4$ and $\mathfrak{C}_4$ are minimal with respect to containing $\mathfrak{N}_4$, each must be the meet of four of the five maximal torsionfree subvarieties of $\mathfrak{N}_5$; and their join, the torsionfree core of $\mathfrak{N}_5 \cap [\mathfrak{A}^2, \mathfrak{E}]$, must be the meet of three. Now $F_2(\mathfrak{N}_5)$ is centre-by-metabelian (because $F_2(\mathfrak{N}_4)$ is metabelian), and if we take $r = 2$ we have $L_5 = L(32) \oplus L(41)$: hence we must have that the torsionfree core of $\mathfrak{N}_5 \cap [\mathfrak{A}^2, \mathfrak{E}]$ is $\mathfrak{N}(21^3) \wedge \mathfrak{N}(2^21) \wedge \mathfrak{N}(31^2)$.

Consider $u(32) = [y,x,x;y,x]$ and $u(41) = [y,x,x,x,x]$. Read as group commutator, $u(32)$ is a law of $(\mathfrak{N}_5 \cap \mathfrak{A}^2) \vee \mathfrak{N}_4$. If it were also a law of $\mathfrak{C}_4$, it would be a law of their join which is the torsionfree core of $\mathfrak{N}_5 \cap [\mathfrak{A}^2, \mathfrak{E}]$, and therefore a law of $F_2(\mathfrak{N}_5)$. Since that is not so, $u(32)$ is not a law of $\mathfrak{C}_4$. Similarly, $u(41)$ is a law of $\mathfrak{C}_4$ but not of $(\mathfrak{N}_5 \cap \mathfrak{A}^2) \vee \mathfrak{N}_4$. It follows that $u(32)$ and $u(41)$ are in disjoint isolated verbal subgroups of $F_2(\mathfrak{N}_5)$. The correspondence which underlies the lattice isomorphism is good enough to yield from this that, read as Lie elements, $u(32)$ and $u(41)$ are in disjoint $\mathrm{GL}(2, \mathbb{Q})$-submodules of L_5 with $r = 2$: but then the only pair of disjoint (nonzero) submodules of L_5 is $L(32)$, $L(41)$. In fact, $u(32) \in L(32)$ and $u(41) \in L(41)$. This can be seen by observing that $[y,x,x,x,x] = yx^4 - 4xyx^3 + 6x^2yx^2 - 4x^3yx + 4x^4y$; that the five (associative) monomials on the right-hand side form a single S_5-orbit; and the span of that orbit, being the direct sum of a 1-dimensional trivial (of type 5) and an irreducible of type 41, avoids $A(32)$: so $u(41)$ could not lie in $L(32)$. Thus we have proved that, *as torsionfree varieties*, $\mathfrak{N}(32)$ *and* $\mathfrak{N}(41)$ *are defined by* $u(32)$ *and* $u(41)$, *respectively* (*the class* 5 *law being, of course, understood*).

So far the laws came easily, but to find those needed to define $\mathfrak{N}(31^2)$ and $\mathfrak{N}(2^21)$ we must work a little harder. It will be convenient to change our field of coefficients from the rational field $\mathbb{Q}$ to the complex field $\mathbb{C}$, specify that A have just three variables, x, y, z, and choose L with $r = 3$. What we are looking for are nonzero polynomials $u(31^2)$ and $u(2^21)$, with integer coefficients, in $L(31^2)$ and $L(2^21)$, respectively. The general linear group $\mathrm{GL}(3, \mathbb{C})$ acts not only on L but on all of A, and its action on A_c is closely linked up with the action of S_c. Each $A(\pi)$ is the sum of all irreducible $\mathrm{GL}(3, \mathbb{C})$-submodules of one isomorphism type, labelled, of course, also by π. As one changes from S_c action to $\mathrm{GL}(3, \mathbb{C})$ action, dimensions and multiplicities swap roles. For instance, we noted that in this 3-variable case $A(2^21)$ is the direct sum of 3 irreducible 5-dimensional S_5-modules: it is also the direct sum of 5 irreducible 3-dimensional $\mathrm{GL}(3, \mathbb{C})$-modules. (This is a special case of a well-known theory, much used by physicists, which originated in the 1901 dissertation [28] of Schur, under whom B. H. Neumann took his first doctorate in Berlin.) It follows that the irreducible $L(2^21)$ has dimension 3; similarly, one obtains that $L(31^2)$ is 6-dimensional, while the dimensions of $L(32)$ and $L(41)$ are 15 and 24, respectively.

It will be useful if we can establish the structure of the $\mathrm{GL}(3, \mathbb{C})$-submodule $[L_2, L_2]$ of L_4. By analogy with the fact that $[L_1, L_1] = L_2$ and the basis x, y, z of L_1 gives rise to the basis $[y,x]$, $[z,x]$, $[z,y]$ of L_2, one sees that $[L_2, L_2]$ has a basis consisting of $[z,x;y,x]$, $[z,y;y,x]$, and $[z,y;z,x]$. If the matrix expressing the action of an element g of $\mathrm{GL}(3, \mathbb{C})$ on L_1 in terms of the given basis is upper triangular, so is the matrix expressing the action of g on $[L_2, L_2]$ in terms of the given basis of that. (Indeed, if zg is a scalar multiple of z and yg a linear combination of y and z, then $[z,y]g$ is a scalar multiple of $[z,y]$ and $[z,x]g$ a linear combination of $[z,x]$ and $[z,y]$, so $[z,y;z,x]g$ is a scalar multiple of $[z,y;z,x]$, and so on.) Moreover, the diagonal entries of the second matrix are obtained from those of the first on multiplication by the determinant det g (of the first matrix). Since every element of $\mathrm{GL}(3, \mathbb{C})$ is conjugate to an 'upper triangular' element (Jordan normal form), this information is sufficient to determine that the character of $\mathrm{GL}(3, \mathbb{C})$ afforded by $[L_2, L_2]$ is the same as the character afforded by $D \otimes L_1$ where D is a 1-dimensional module spanned, say, by d, such that $dg = (\det g)d$ for every g in $\mathrm{GL}(3, \mathbb{C})$. As $D \otimes L_1$ is clearly irreducible, this implies that $D \otimes L_1 \cong [L_2, L_2]$. In fact, we shall need to specify an isomorphism. The unique l-dimensional subspaces of L_1 and $[L_2,L_2]$ invariant under the 'upper triangular' subgroup of $\mathrm{GL}(3, \mathbb{C})$ were spanned by z and $[z,y;z,x]$, respectively, so an isomorphism must map $d \otimes z$ to a nonzero scalar multiple of $[z,y;z,x]$; as a (nonzero) scalar multiple of an isomorphism is an isomorphism, we conclude that there is an isomorphism which maps $d \otimes z$ to $[z,y;z,x]$. Now we make use of the elements of $\mathrm{GL}(3, \mathbb{C})$ which permute the three variables to deduce that this isomorphism must map $d \otimes x$ to $-[z,x;y,x]$ and $d \otimes y$ to $-[z,y;y,x]$.

The purpose of this preparation was to enable us to exploit the well-known fact that $L_1 \otimes L_1$ is the direct sum of the 3-dimensional space of skew-symmetric tensors and the 6-dimensional space of symmetric tensors, these two being irreducible $\mathrm{GL}(3, \mathbb{C})$-modules. As $[L_2, L_2] \otimes L_1$ is isomorphic to $D \otimes L_1 \otimes L_1$, this means that $[L_2, L_2] \otimes L_1$ is the direct sum of two irreducibles, a 3-dimensional containing $[z,y;z,x] \otimes y + [z,y;y,x] \otimes z$ and a 6-dimensional containing $[z,y;z,x] \otimes z$. Now $a \otimes b \mapsto [a,b]$ is a homomorphism of $[L_2,L_2] \otimes L_1$ onto the submodule $[L_2, L_2, L_1]$ of L_5, so we conclude that $[z,y;z,x;y] + [z,y;y,x;z]$ and $[z,y;z,x;z]$ generate a

3-dimensional and a 6-dimensional irreducible submodule of L_5, respectively; that is, unless one or both vanish. Having calculated the dimensions of the irreducible submodules of L_5, we know this means that the first element lies in $L(2^21)$ and the second in $L(31^2)$. To verify that neither element is zero, one converts to basic form using the Jacobi identity. The result is, in group notation,

$$u(2^21) = [z,y,y;z,x][z,x,y;z,y]^{-1}[z,y,z;y,x][y,x,z;z,y]^{-1},$$

and $u(31^2) = [z,x,z;z,y]^{-1}[z,y,z;z,x]$. *The varieties defined by $u(2^21)$ and $u(31^2)$ as torsionfree varieties, are then $\mathfrak{N}(2^21)$ and $\mathfrak{N}(31^2)$, respectively (the class 5 law being again understood).* This completes the search for defining laws.

It remains to consider further identifications in terms other than explicit defining laws. Two general results are of use here; each is an interpretation of the fact that the number of summands in an unrefinable direct decomposition of $L(\pi)$ as $\mathrm{GL}(r, \mathbb{Q})$-module is the multiplicity $l(\pi)$ discussed in [15], independent of r as long as r is at least the number of parts of π, while $L(\pi) = 0$ for all smaller values of r. The first asserts that if $\mathfrak{B}$ is a torsionfree variety between $\mathfrak{N}_{c-1}$ and $\mathfrak{N}_c$, then the join of $\mathfrak{N}_{c-1}$ with the variety generated by the rank r free group $F_r(\mathfrak{B})$ of $\mathfrak{B}$ is just the meet of $\mathfrak{B}$ with the variety

$$\wedge \{\mathfrak{N}(\pi) \mid \pi \text{ has more than } r \text{ parts}\}.$$

The second is that if $\mathfrak{B} = \wedge \{\mathfrak{N}(\pi) \mid \pi \in \Pi\}$ (note not every $\mathfrak{B}$ is of this form when $c > 5$), then the torsionfree core of $\mathfrak{N}_c \cap \mathfrak{B}^{(r)}$ is just

$$\wedge \{\mathfrak{N}(\pi) \mid \pi \in \Pi \text{ and } \pi \text{ has at most } r \text{ parts}\}.$$

(There is a finer version of this, to cope with all $\mathfrak{B}$ even when $c > 5$, but to state that would require finer terminology.) We have already noted that $\mathfrak{N}(21^3)$ is generated by $F_3(\mathfrak{N}_5)$, and that the torsionfree core of $\mathfrak{N}_5 \cap [\mathfrak{A}^2, \mathfrak{E}]$, generated by $F_2(\mathfrak{N}_5)$, is $\mathfrak{N}(21^3) \wedge \mathfrak{N}(2^21) \wedge \mathfrak{N}(31^2)$. The torsionfree core of $\mathfrak{N}_4^{(3)}$ (which lies in $\mathfrak{N}_5$ on account of Theorem B of Newman [25]) is $\mathfrak{N}(2^21) \wedge \mathfrak{N}(31^2) \wedge \mathfrak{N}(32) \wedge \mathfrak{N}(41)$, while that of $\mathfrak{N}_5 \cap \mathfrak{N}_4^{(2)}$ is $\mathfrak{N}(32) \wedge \mathfrak{N}(41)$. Using one method after another, one obtains that $\mathfrak{N}(2^21) \wedge \mathfrak{N}(31^2)$ is the torsionfree core of $\mathfrak{N}_5 \cap [\mathfrak{A}^2, \mathfrak{E}]^{(3)}$, and that $\mathfrak{N}(21^3) \wedge \mathfrak{N}(32) \wedge \mathfrak{N}(41)$ is generated by the rank 3 free group of the torsionfree core of $\mathfrak{N}_5 \cap \mathfrak{N}_4^{(2)}$. The reader may care to complete the list obtainable in this way; I shall merely add that $\mathfrak{N}(32)$ is the torsionfree core of $\mathfrak{N}_5 \cap (\mathfrak{A}^2)^{(2)}$, that is, $\mathfrak{N}(32)$ is generated by the torsionfree groups of class at most 5 whose 2-generator subgroups are all metabelian. Two identifications which do not come simply by these methods may also be noted: $\mathfrak{N}(21^3) \wedge \mathfrak{N}(2^21) \wedge \mathfrak{N}(31^2) \wedge \mathfrak{N}(32)$ is not only $(\mathfrak{N}_5 \cap \mathfrak{A}^2) \vee \mathfrak{N}_4$ but also the torsionfree core of $\mathfrak{N}_5 \cap [\mathfrak{N}_2, \mathfrak{A}]$, while $\mathfrak{N}(41)$ is not only the torsionfree core of $\mathfrak{N}_5 \cap \mathfrak{E}_4^{(2)}$ but also the variety generated by the torsionfree fourth Engel groups of class at most 5.

The end result is that we have, or can generate, at least one such identification for each of the 14 torsionfree varieties strictly between $\mathfrak{N}_4$ and $\mathfrak{N}_5$ whose meet decompositions involve *either* both $\mathfrak{N}(2^21)$ and $\mathfrak{N}(31^2)$ *or* neither. However, I cannot envisage any way of reaching the other 16 in such terms.

5. Applications

There are at least two reasons why detailed knowledge of torsionfree varieties is not only of intrinsic interest but is capable of powerful application in the context of other varieties.

The first is that many a variety is the join of its torsionfree core and another, significantly smaller variety. For instance, every nilpotent variety is the join of its torsionfree core and a finite exponent variety (proved like 2.5 is in Newman [25]). Another way of putting this is that a join-irreducible nilpotent variety is either torsionfree or of finite exponent. A similar fact is that a join-irreducible metabelian variety is either torsionfree or of finite exponent or of the form $\mathfrak{A}_{p^k}\mathfrak{A}$ for some prime power p^k (Kovács and Newman, see Bryce [3]). This is not the place to attempt it, but a survey of results of this kind would be an interesting project, and a systematic search for further theorems could well prove fruitful.

The second is that one can exploit the torsionfree subvarieties of a variety $\mathfrak{V}$ in the description of its finite exponent subvarieties. In a sense, small class theory already illustrates this, but I mean something more direct here. The ideal, but perhaps misleadingly simple (I don't mean easy) case is that of $\mathfrak{V} = \mathfrak{N}_c \cap \mathfrak{A}^2$: for $p > c$, the classification of the p-power-exponent subvarieties of $\mathfrak{V}$ by Brisley [2] and Weichsel [31] allows such an interpretation. Let me sketch, rather speculatively, how a more general application might run. Let $\mathfrak{V} \leq \mathfrak{N}_c$ and $p > c$. It is easy to prove that if $\mathfrak{U} < \mathfrak{W}$ in $\Lambda^0(\mathfrak{V})$ then $\mathfrak{B}_p \cap \mathfrak{U} < \mathfrak{B}_p \cap \mathfrak{W}$. Progress is comfortable if $\mathfrak{U} \mapsto \mathfrak{B}_p \cap \mathfrak{U}$ is a lattice homomorphism from $\Lambda^0(\mathfrak{V})$ into $\Lambda(\mathfrak{B}_p \cap \mathfrak{V})$: this happens if and only if $\mathfrak{U} \cap \mathfrak{W}$ is p-torsionfree (that is, its free groups have no elements of order p) for all $\mathfrak{U}, \mathfrak{W}$ in $\Lambda^0(\mathfrak{V})$. By the previous comment, in this case the lattice homomorphism is one-to-one, hence $\Lambda^0(\mathfrak{V})$ is finite and, by a result proved in Section 2, distributive. It can then be deduced from small class theory that the lattice $\Lambda^p(\mathfrak{V})$ of all p-power-exponent subvarieties of $\mathfrak{V}$ is also distributive. If in addition $\mathfrak{U} \mapsto \mathfrak{B}_p \cap \mathfrak{U}$ maps $\Lambda^0(\mathfrak{V})$ *onto* $\Lambda(\mathfrak{B}_p \cap \mathfrak{V})$, one should be able to obtain a full description of $\Lambda^p(\mathfrak{V})$ from $\Lambda^0(\mathfrak{V})$. Such descriptions of distributive lattices with minimum condition are conveniently given, as has been the practice in variety theory, in terms of the partially ordered set of the join-irreducible elements of the lattice. The join-irreducibles of $\Lambda^p(\mathfrak{V})$ should be just the varieties of the form $\mathfrak{B}_{p^k} \cap \mathfrak{U}$ with $\mathfrak{U}$ join-irreducible in $\Lambda^0(\mathfrak{V})$, and their partial order should be given by $\mathfrak{B}_{p^{k(1)}} \cap \mathfrak{U}_1 \leq \mathfrak{B}_{p^{k(2)}} \cap \mathfrak{U}_2$ if and only if $k(1) \leq k(2)$ and $\mathfrak{U}_1 \leq \mathfrak{U}_2$. I expect that this sketch works when $\mathfrak{V}$ is either $\mathfrak{N}_5$ or any $\mathfrak{N}_c \cap [\mathfrak{A}^2, \mathfrak{E}]$, but I have not checked all the details. It would be interesting to pin down the conditions under which it works in general. The reason for including it is to indicate that finite and distributive $\Lambda^0(\mathfrak{V})$ are not only the most manageable but also the most applicable sets of torsionfree varieties, and so deserve the special attention directed to them in the present paper and particularly in our open problems.

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