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A problem of Hanna Neumann on closed sets of group words

L.G. Kovács and M.R. Vaughan-Lee

In Problem 1 of her book *Varieties of groups*, Hanna Neumann asked whether a fully invariant subsemigroup of a free group of infinite rank is necessarily a subgroup. This note presents an example which shows that the answer is negative.

Notation and terminology follow Hanna Neumann's book [1].

Let $\{g, h\}$ be a free generating set of the free group G of rank 2 in the variety $\underline{\underline{N}}_6$ of all nilpotent groups of class at most 6, and let $u = \left[[h, g, g, g], [h, g]\right]$. Note that, with the obvious order on the given free generating set, u is a basic commutator. A routine calculation shows that if the image of u under an arbitrary endomorphism of G is expressed in terms of basic commutators, in this expression u itself will occur with square exponent (and, of course, only commutators of weight 6 occur with nonzero exponent). Consequently, in the basic commutator expression of a product of endomorphic images of u the exponent of u is nonnegative, and so u^{-1} is not such a product.

It follows that if $v = \left[[x_2, x_1, x_1, x_1], [x_2, x_1]\right]$ in X_∞, then v^{-1} does not lie in the (fully invariant) subsemigroup of X_∞ generated by the images of v under the endomorphisms of X_∞. This answers Problem 1 of Hanna Neumann's book [1] in the negative.

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variant of this example settles a question which had been put to him by Professor Graham Higman in July 1958. Namely, let H denote the factor group of G over the (central) subgroup generated by the basic commutators of weight 6 other than u, and let h denote the image of u in H: then all values of v in H are of the form h^{n^2}. As a finitely generated torsionfree nilpotent group, H can be fully ordered; do this so that $h > 1$. Now $v\phi \geq 1$ for every value $v\phi$ of v in H, and of course $v\phi = h \neq 1$ for a suitable substitution ϕ. The question was whether any word could be nontrivially semi-definite on any ordered group, in the sense in which v is on H.

Reference

[1] Hanna Neumann, *Varieties of groups* (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

Institute of Advances Studies,
Australian National University,
Canberra, ACT,

and

Department of Mathematics,
University of Queensland,
St Lucia, Queensland.