THE DESCENDING CHAIN CONDITION IN JOIN-CONTINUOUS MODULAR LATTICES

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If L is a distributive lattice in which every element is the join of finitely many join-irreducible elements, and if the set of join-irreducible elements of L satisfies the descending chain condition, then L satisfies the descending chain condition: this follows easily from the results of Chapter VIII, Section 2, in the Third (New) Edition of Garrett Birkhoff's *'Lattice Theory'* (Amer. Math. Soc., Providence, 1967). Certain investigations (M. S. Brooks, R. A. Bryce, unpublished) on the lattice of all subvarieties of some variety of algebraic systems require a similar result without the assumption of distributivity. Such a lattice is always *join-continuous*: that is, it is complete and $(\bigwedge X) \vee y = \bigwedge \{x \vee y : x \in X\}$ whenever X is a chain in the lattice (for, the dual of such a lattice is complete and 'algebraic', in Birkhoff's terminology). The purpose of this note is to present the result:

THEOREM. *Let L be a join-continuous modular lattice. The descending chain condition is satisfied by L if (and obviously only if)*

(i) *every element of L is a join of finitely many join-irreducible elements, and*

(ii) *the set M of join-irreducible elements of L satisfies the descending chain condition.*

It would be interesting to know whether this remains a theorem if the assumption of modularity (and/or of join-continuity) is omitted.

Use will be made of a lemma, which states what the proof of Theorem 2 of Birkhoff (loc. cit.) shows; however, it will be established here by an apparently simpler argument.

LEMMA. *Let M be a partially ordered set satisfying the descending chain condition, and let $\mathcal{N}$ be the set of those finite subsets of M which consist of mutually incomparable elements. Define a partial order $\leqq$ on $\mathcal{N}$ by putting*

$$A \leqq B \quad \textit{if} \quad \forall a \in A \,.\, \exists b \in B \,.\, a \leqq b.$$

Then $\mathcal{N}$ satisfies the descending chain condition.

PROOF. It is easy to check that the relation $\leqq$ defined on $\mathscr{N}$ is indeed a partial order. Suppose the Lemma is false, and

$$A_1 > \cdots > A_i > \cdots$$

is an infinite properly descending chain (of type ω) in $\mathscr{N}$. Then $\bigcup_i A_i$ is infinite. Consider the sequences

$$\text{(a)} \qquad a_1 \geqq \cdots \geqq a_i \geqq \cdots \qquad a_i \in A_i$$

which are maximal: that is, either infinite, or finite with last term a_n such that A_{n+1} has no element a_{n+1} with $a_n \geqq a_{n+1}$. As each element of the infinite set $\bigcup_i A_i$ occurs in some such sequence while each sequence has only finitely many distinct terms, there must be infinitely many such sequences. Given a positive integer k, there are only finitely many (not necessarily maximal) sequences of length k which can occur as initial segments of the sequences (a): thus at least one sequence of length k, say

$$b_1 \geqq \cdots \geqq b_k \qquad b_i \in A_i,$$

is the initial segment of infinitely many sequences (a). Of these, infinitely many must have the same initial segment of length $k+1$, say

$$b_1 \geqq \cdots \geqq b_k \geqq b_{k+1} \qquad b_i \in A_i.$$

Inductively, one obtains the existence of an infinite sequence

$$\text{(b)} \qquad b_1 \geqq \cdots \geqq b_k \geqq \cdots \qquad b_i \in A_i$$

such that each initial segment of (b) is also the initial segment of infinitely many other (maximal) sequences. Now (b) must be constant from some term on: say, $b_m = b_{m+1} = \cdots$. Let (a) be another sequence with initial segment $b_1 \geqq \cdots \geqq b_m$; that is, with $a_1 = b_1, \cdots, a_m = b_m$. As (a) is maximal, it cannot be an initial segment of (b); hence there will be an integer n with $a_{m+n} \neq b_{m+n}$, but of course with

$$a_{m+n} \leqq a_m = b_m = b_{m+n}:$$

so that $a_{m+n} < b_{m+n}$, contrary to the fact that A_{m+n} consists of mutually incomparable elements. This contradiction completes the proof.

PROOF OF THE THEOREM. Suppose that $x_1 \geqq \cdots \geqq x_i \geqq \cdots$ is a descending chain (of type ω) in L, and put $x = \bigwedge_i x_i$. The first step is to show that the dual ideal D generated by x also satisfies the hypotheses: the rest of the argument can be carried out in D, or, still more conveniently, it can be assumed without loss of generality that x is the least element of L.

Obviously, D is modular and join-continuous. It also inherits (i), for $y \to x \vee y$ is a join-homomorphism of L onto D which maps join-irreducibles to join-irreducibles: if a is join-irreducible in L, it is certainly join-irreducible in the interval $[x \wedge a, a]$, and so — by the isomorphism theorem of modular lattices — $x \vee a$ is join-irreducible in $[x, x \vee a]$ and hence also in D. Suppose that $d_1 \geqq \cdots \geqq d_i \geqq \cdots$ is a descending chain of join-irreducible elements of D. Write d_1 as a join of join-irreducibles $a_1, \cdots, a_m$ of L; then d_1 is also the join of their images in D, and hence one of these images is d_1: say, $d_1 = x \vee a_1$. Put $d'_1 = a_1$. Next, suppose that $d_i = x \vee d'_i$ with d'_i join-irreducible in L. Then, as $d_i \geqq d_{i+1} \geqq x$ and L is modular, $d_{i+1} = x \vee (d_{i+1} \wedge d'_i)$. Write $d_{i+1} \wedge d'_i$ as a join of join-irreducibles of L: say, of $b_1, \cdots, b_n$. As d_{i+1} is the join of their images in D, one such image must be d_{i+1} itself: say, $d_{i+1} = x \vee b_1$. Put $d'_{i+1} = b_1$; note that $d'_i \geqq d'_{i+1}$. Inductively, it is possible to select a descending chain $d'_1 \geqq \cdots \geqq d'_i \geqq \cdots$ of join-irreducibles of L such that

$$d_1 = x \vee d'_1, \cdots, d_i = x \vee d'_i, \cdots.$$

As M satisfies the descending chain condition, from some term on $d'_k = d'_{k+1} = \cdots$, and hence $d_k = d_{k+1} = \cdots$. This proves that D inherits (ii).

From now on it will be assumed that x is the least element of L.

Let $\mathscr{F}$ be the set of all finite subsets of L, quasi-ordered by the relation

$$A \leqq B \quad \text{if} \quad \forall a \in A. \quad \exists b \in B \,.\, a \leqq b.$$

Let $\mathscr{I}$ be the set of all those finite non-empty subsets J of M which give their joins irredundantly: that is, if $a \in J$ then either $\bigvee J \neq \bigvee (J \setminus \{a\})$ or $J = \{x\}$. (The join of the empty subset of L is interpreted as the least element of L.) Note that $\mathscr{I}$ is contained in $\mathscr{N}$ which in turn is contained in $\mathscr{F}$, and the partial order of $\mathscr{N}$ is just the restriction of the quasi-order of $\mathscr{F}$. By the Lemma, $\mathscr{I}$ satisfies the descending chain condition with respect to this partial order $\leqq$. Moreover, it is an easy consequence of (i) that

$$(*) \qquad \forall A \in \mathscr{F} \,.\, \exists J \in \mathscr{I} \,.\, J \leqq A \;\&\; \bigvee J = \bigvee A.$$

Let y be any element of L, and J a minimal element of the set $\{J \in \mathscr{I} : y \leqq \bigvee J\}$ (note that, on account of (i), this set cannot be empty). The next step is to show that if $a \in J$ and $J' = J \setminus \{a\}$ then $\bigvee J = (\bigvee J') \vee y$. To this end, consider $a^* = a \wedge ((\bigvee J') \vee y)$ and $A = J' \cup \{a^*\}$. By construction, $A \leqq J$. By the modular law,

$$\begin{aligned} \bigvee A &= (\bigvee J') \vee a^* = (\bigvee J') \vee (a \wedge ((\bigvee J') \vee y)) \\ &= ((\bigvee J') \vee a) \wedge ((\bigvee J') \vee y) \\ &= (\bigvee J) \wedge ((\bigvee J') \vee y) = (\bigvee J') \vee y \geqq y. \end{aligned}$$

According to $(*)$, $\exists J^* \in \mathscr{I} \,.\, J^* \leqq A \leqq J \;\&\; \bigvee J^* = \bigvee A \geqq y$. The minimal choice of J now implies that $J^* = J$, thus $\bigvee J = \bigvee A$, and it has already been shown that $\bigvee A = (\bigvee J') \vee y$.

For the final step, let J_1 be a minimal element of the set $\{J \in \mathscr{I} : x_1 \leqq \bigvee J\}$. If J_i has already been chosen so that $x_i \leqq \bigvee J_i$, then the set $\{J \in \mathscr{I} : x_{i+1} \leqq \bigvee J \;\&\; J \leqq J_i\}$ is non-empty; choose J_{i+1} as a minimal element from it. Inductively one obtains a descending chain $J_1 \geqq \cdots \geqq J_i \geqq \cdots$ in $\mathscr{I}$ such that $x_i \leqq \bigvee J_i$ and, if $a_i \in J_i$, $J'_i = J_i \setminus \{a_i\}$, then $(\bigvee J'_i) \vee x_i = \bigvee J_i$. As $\mathscr{I}$ satisfies the descending chain condition, $J_m = J_{m+n}$ for some m and every n; now it is possible to choose $a_{m+1}, \cdots, a_{m+n}, \cdots$ all equal to a_m, so that $J'_m = \cdots = J'_{m+n} = \cdots$, and then $\bigvee J_m = (\bigvee J'_m) \vee x_{m+n}$ for every n. Put $X = \{x_m, \cdots, x_{m+n}, \cdots\}$ and use that L is join-continuous:

$$(\bigvee J'_m) \vee x = (\bigvee J'_m) \vee (\bigwedge X) = \bigwedge_n ((\bigvee J'_m) \vee x_{m+n}) = \bigvee J_m.$$

Since x is the least element of L, this means that $\bigvee J'_m = \bigvee J_m$. As J_m gives its join irredundantly, this can only happen if $J_m = \{x\}$. Thus $x = \bigvee J_m \geqq x_m$ yields that $x_m = \cdots = x_{m+n} = \cdots = x$, and the proof is complete.

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