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Varieties and the Hall–Higman paper

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A class $\mathfrak{X}$ of groups is said to have the Burnside property if $\mathfrak{X}$ consists of finite groups and, for each positive integer k, $\mathfrak{X}$ contains only finitely many (isomorphism classes of) k-generator groups. For each positive integer e, let $\mathfrak{R}_e$ denote the class of finite groups of exponent dividing e, and $\mathfrak{S}_e$ the class of soluble groups in $\mathfrak{R}_e$. The restricted Burnside conjecture (R_e) for exponent e is that $\mathfrak{R}_e$ has the Burnside property, and the corresponding soluble Burnside conjecture (S_e) asserts the same for $\mathfrak{S}_e$. A group will be called simply monolithic if the intersection of its nontrivial normal subgroups is a nonabelian simple group. The factors of a group are the factor groups of its subgroups.

P. Hall and G. Higman [1] gave some reduction theorems for the Burnside conjectures and remarked (p. 39 in [1]) that "theorems of absolute validity" could also be deduced by their method. The purpose of this note is to put on record some facts about varieties which follow from, or are closely related to, the paper of Hall and Higman. The following statement appears to express the full force of their method:

(A) *If $\mathfrak{X}$ is a class with the Burnside property and $\mathfrak{Y}$ is a class of finite groups whose Sylow subgroups and simply monolithic factors all lie in $\mathfrak{X}$, then $\mathfrak{Y}$ also has the Burnside property.*

One further observation will be relevant; this relies on more recent developments. It can be checked that each class $\mathfrak{R}_e$ contains only a finite number of the simple groups which are known at present (in fact, each large number which is known to occur as the order of a finite simple group is divisible by some large prime). Let an A^*-group be a locally finite group whose Sylow subgroups are all abelian. According to Z. Janko, there exist now (mostly unpublished) results (due to D. Gorenstein, Z. Janko, J. G. Thompson, J. H. Walter, and N. Ward) which show that all finite simple A^*-groups are known. These facts and (A) together imply:

(B) *For each positive integer e, the class of finite A^*-groups of exponent dividing e has the Burnside property.*

For a class $\mathfrak{X}$ of groups, let $\mathfrak{X}_0$ denote the class of the finitely generated subgroups of the groups in $\mathfrak{X}$. (The variety var $\mathfrak{X}$ generated by $\mathfrak{X}$ is obviously the same as var $\mathfrak{X}_0$.) In a conversation with M. F. Newman it was observed that var $\mathfrak{X}$ is locally finite if and only if $\mathfrak{X}_0$ has the Burnside property. Thus (R_e) is equivalent to the statement that the class of all locally finite groups of exponent dividing e is a variety, and (S_e) is equivalent to: "the class of all locally finite-and-soluble groups of exponent dividing e is a variety." In particular, A. I. Kostrikin's result [2] that (R_p) is true for every prime p means that the class of all locally finite groups of exponent dividing p is a variety: call it the Kostrikin variety $\mathfrak{K}_p$ of exponent p.

The first of the four facts to be mentioned here concerns a special case of the finite basis problem. From Schreier's Theorem it follows easily that the laws of a locally finite variety $\mathfrak{V}$ are finitely based if and only if $\mathfrak{V}$ can be defined by its k-variable laws for some positive integer k. Thus

(1) *The laws of $\mathfrak{K}_p$ are not finitely based if and only if there exist, to each positive integer k, infinite $(k+1)$-generator groups of exponent p in which all k-generator subgroups are finite.*

(2) *If $\mathfrak{V}$ is a locally finite variety, then the locally soluble groups in $\mathfrak{V}$ form a subvariety $\mathfrak{V}_{LS}$, and the locally nilpotent groups form a subvariety $\mathfrak{V}_{LN}$. If $\mathfrak{V}_{LN}$ is soluble, so is $\mathfrak{V}_{LS}$.*

The proof of the first statement is straight-forward; the second follows from Theorem 3.6.2 of [1].

(3) *If $\mathfrak{U}$ is a locally finite variety and $\mathfrak{V}$ is the class of those groups whose nilpotent factors and finitely generated simply monolithic factors all belong to $\mathfrak{U}$, then $\mathfrak{V}$ is also a locally finite variety.*

This is proved from (A). Similarly, one obtains from (B) the following:

(4) *For each positive integer e, the class $\mathfrak{A}_e^*$ of A^*-groups of exponent dividing e is a (locally finite) variety.*

References

[1] P. Hall and G. Higman, On the p-length of p-soluble groups and reduction theorems for Burnside's problem, *Proc. London Math. Soc.* (3) **6** (1956), 1–42.

[2] A. I. Kostrikin, On Burnside's problem, *Izv. Akad. Nauk SSSR Ser. Mat.* **23** (1959), 3–34 [Russian].