INVOLUTORY AUTOMORPHISMS OF GROUPS OF ODD ORDER AND THEIR FIXED POINT GROUPS

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In Memoriam Tadasi Nakayama

1. **Introduction.** Let G be a finite group of odd order with an automorphism θ of order 2. (We use without further reference the fact, established by W. Feit and J. G. Thompson, that all groups of odd order are soluble.) Let G_θ denote the subgroup of G formed by the elements fixed under θ. It is an elementary result that if $G_\theta = 1$ then G is abelian. But if we merely postulate that G_θ be cyclic, the structure of G may be considerably more complicated—indeed G may have arbitrarily large soluble length. E.g. let p be an odd prime and let t denote the largest odd divisor of $p-1$. Let G be the group formed by the matrices $A = \begin{pmatrix} u+pa & pb \\ pc & v+pd \end{pmatrix}$ of determinant 1, where a, b, c, d, u, v lie in the ring of residue classes (mod p^{k+1}) and $uv \equiv u^t \equiv 1 \pmod{p}$. Let θ be the contragradient automorphism $A \to (A^{-1})^T$. Then G_θ is cyclic of order p^k. G itself has order $p^{3k}t$ and soluble length m or $m+1$, where m is the least integer such that $2^m \geq k+1$. The Fitting subgroup of G is a p-group of order p^{3k}, exponent p^k, and class k.

The theorem proved in this note deals with the case where G_θ is nilpotent. It belongs to the same circle of ideas as the recent results of J. G. Thompson [5], though it is much more special. Let $F(H)$ denote the Fitting subgroup of a group H and $1 = F_0(H) \leq F_1(H) \leq \cdots$ the ascending Fitting series of H, defined inductively by $F_{i+1}(H)/F_i(H) = F(G/F_i(H))$.

Theorem Let G be a group of odd order and θ an automorphism of G of order 2 such that G_θ is nilpotent. Suppose either (i) $G = F_2(G)$ or (ii) the Sylow subgroups of G_θ are regular. Then $G/F(G)$ is contained in the variety $\mathbf{V}$ generated by G_θ together with the cyclic subgroups of $G/F(G)$.

The following particular case may serve as an illustration of the Theorem:

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if G is a group of odd order having an automorphism of order 2 with abelian fixed point group, then G' is nilpotent. The example cited above shows that, even in this case, $G/F(G)$ need not belong to the variety generated by G_θ.

Conditions (i), (ii) in the Theorem are not superfluous, i.e. the nilpotency of G_θ does not imply that $G=F_2(G)$. To see this, let Γ be the group, of order $2.3^7.7$, formed by the transformations $x \to ax^\alpha + b$ of the field $K=GF(3^6)$, where a runs over the 7-th roots of unity in K, α over the automorphisms of K and b over all the elements of K. Let G be the subgroup of Γ of index 2 and θ the automorphism $X \to T^{-1}XT$ of G, where T is the transformation $x \to x^{27}$. Then G_θ is a (non-regular) 3-group whereas $F_2(G)$ is a subgroup of G of index 3.

On the other hand, the nilpotency of G_θ implies, by the results of Thompson [5], that $G' \leq F_3(G)$ and thus that $G=F_4(G)$. J. N. Ward has proved [6] that, in fact, the nilpotency of G_θ implies $G=F_3(G)$.

We mention in conclusion another unpublished result of Ward dealing with the derived length of $G/F(G)$: if G_θ is nilpotent and $(G_\theta)^{(n)}=1$, then $G^{(n)}$ is nilpotent. The point of Ward's result is that it deals with the case where $G/F(G)$ is not necessarily nilpotent.

2. Notation. Our general notation is standard and only the following points need to be mentioned. $M \leq N$ ($M \trianglelefteq N$) means that M is a subgroup (normal subgroup) of the group N. $M<N$ ($M \triangleleft N$) means that M is a subgroup (normal subgroup) of N distinct from N. If $L \trianglelefteq M \leq N$, M/L is called a factor of N.

Throughout the paper, G denotes a group of odd order and θ an automorphism of G of order ≤ 2 such that G_θ is nilpotent. Γ denotes the splitting extension of G by θ. $\mathbf{V}$ denotes the variety generated by G_θ together with the cyclic subgroups of $G/F(G)$; note that every group in $\mathbf{V}$ is nilpotent. If H is a group, $H/\mathbf{V}(H)$ denotes the largest factor group of H belonging to $\mathbf{V}$. As is well known, $\mathbf{V}(K) \leq \mathbf{V}(H)$ if $K \leq H$, $\mathbf{V}(H/N)=\mathbf{V}(H)N/N$ if $N \trianglelefteq H$, and $\mathbf{V}(H \times K) = \mathbf{V}(H) \times \mathbf{V}(K)$.

A subgroup H of G is called a θ-subgroup if $H=H^\theta$. A factor H/K of G is called a θ-factor if H, K are θ-subgroups. The automorphism θ acts in a natural way as an automorphism of order ≤ 2 on each θ-factor H/K. The subgroup formed by those elements of H/K which are fixed under θ is denoted by $(H/K)_\theta$.

LEMMA 1. *Let H/K be a θ-factor of G and xK an element of H/K such* *that* $(xK)^{\theta}=(xK)^{\varepsilon}$ $(\varepsilon=\pm 1)$. *Then* xK *contains an element* y *such that* $y^{\theta}=y^{\varepsilon}$.

Proof. The mapping $u\to(u^{\varepsilon})^{\theta}$ is a permutation of order ≤ 2 of the set xK of odd order; take y as any of the elements fixed under this permutation.

COROLLARY. $(H/K)_{\theta}=H_{0}K/K$; *thus* $(H/K)_{\theta}$ *is isomorphic to a factor of* G_{θ}.

3. The present section leads up to the proof of the Theorem in Section 4. We derive some properties of the Fitting series of G under the following assumptions:

(3.1) $G/F(G)\notin \mathbf{V}$;

(3.2) if K is a θ-factor of G different from G, then $K/F(K)\in \mathbf{V}$.

Set $F=F(G)$, $F_i=F_i(G)$, and $\Phi=\Phi(G)$, the Frattini subgroup of G.

LEMMA 2. *F is the unique minimal normal subgroup of Γ. F is its own centralizer $C_{\Gamma}(F)$ in Γ.*

Proof. Suppose $F(\Gamma)\neq F$. Then $|F(\Gamma):F|=2$ and the Sylow 2-subgroup of $F(\Gamma)$ (of order 2) is normal in Γ. Thus $\Gamma=gp\{\theta\}\times G$ and so $G=G_{\theta}$, contrary to (3.1). Hence $F(\Gamma)=F$. As Γ is soluble, a result of H. Fitting [2] gives that

(3.3) $C_{\Gamma}(F)=C_{\Gamma}(F(\Gamma))\leq F$.

Let M_1, M_2 be minimal normal subgroups of Γ. Then $M_i\leq F(\Gamma)\leq G$. By (3.2), $\mathbf{V}(G/M_i)$ is nilpotent. Since $G/M_1\cap M_2$ is isomorphic to a subgroup of $G/M_1\times G/M_2$, $\mathbf{V}(G/M_1\cap M_2)$ is also nilpotent; hence, by (3.1), $M_1\cap M_2>1$ and so $M_1=M_2$. It follows that $M_1=M_2=M$ (say) is the unique minimal subgroup of Γ. As Γ is soluble, M is an elementary abelian p-group for some prime p.

By a result of W. Gaschütz [3], $F(G/\Phi)=F/\Phi$. Since $\mathbf{V}(G)\nleq F$ (by (3.1)), also $\mathbf{V}(G/\Phi)\nleq F(G/\Phi)$ and therefore $\Phi=1$ (by (3.2)). As $\Phi(F)\leq\Phi$, F is a direct product of elementary abelian groups. Since M is the unique minimal normal subgroup of Γ, F is in fact an elementary abelian p-group.

Consider $H/M=F(G/M)$. If P/M is the Sylow -subgroup of H/M, then P is a normal p-subgroup of G: hence $P\leq F$. But clearly $F\leq P$, so that $F=P$. Since $\mathbf{V}(G)\leq H$ (by (3.2)) and $\mathbf{V}(G)\nleq F$ (by (3.1)), we have

(3.4) $F<H$.

Now F is a normal, elementary abelian Sylow subgroup of H. Therefore

F is a completely reducible (H/F)-module. It follows that $F=M\times N$ with $N\lhd H$. Since H/M is nilpotent, H/F acts trivially on N. As F is abelian, we deduce that $N\leq Z(H)$. It $Z(H)>1$, then $M\leq Z(H)$ and so $F=M\times N\leq Z(H)$, contrary to (3.3) and (3.4). Thus $Z(H)=1$, $N=1$, $F=M$, which proves the lemma.

Restating the lemma in module terminology, we have

COROLLARY 1. *F is an elementary abelian p-group for some prime p. F is a faithful, irreducible (Γ/F)-module.*

COROLLARY 2. $F_\theta>1$.

Proof. Suppose $F_\theta=1$. Then θ must invert all the elements of F. Since F is a faithful (Γ/F)-module, it follows that $\theta F\in Z(\Gamma/F)$ and so $(G/F)_\theta=G/F$. By the Corollary to Lemma 1, this implies that $G/F\in \mathbf{V}$, contrary to (3.1).

COROLLARY 3. *If x is a p'-element of G_θ (i.e. an element of G_θ of order prime to p), then $C_F(x)\geq F_\theta>1$.*

Proof. The inclusion $C_F(x)\geq F_\theta$ follows directly from the nilpotency of G_θ.

LEMMA 3. *Let K/F be a θ-subgroup of G/F distinct from G/F. Suppose there exists a normal θ-subgroup H/F of G/F such that $\mathbf{V}(K)\leq H\leq K$. Then $\mathbf{V}(K/F)=1$.*

Proof. Since $K=K^\theta<G$, $\mathbf{V}(K)\leq F(K)$ by (3.2). Hence $\mathbf{V}(K)\leq F(K)\cap H$. Since $H\leq K$ and $F(K)\cap H\trianglelefteq K$, we have $F(K)\cap H\trianglelefteq H$ and so $F(K)\cap H\leq F(H)$. On the other hand, since $H\trianglelefteq G$, $F(H)\leq F$. It follows that $\mathbf{V}(K)\leq F(K)\cap H\leq F$ and thus that $\mathbf{V}(K/F)=1$.

COROLLARY. *If H/F is a proper normal θ-subgroup of G/F, then $\mathbf{V}(H/F)=1$.*

LEMMA 4. *F_2/F is the unique minimal normal subgroup of Γ/F. G/F_2 has prime order r and F_2/F is an elementary abelian q-group for some prime $q\neq p$ or r. $Z(G/F)=1$.*

Proof. By the last Corollary, every proper normal θ-subgroup of G/F is nilpotent. Hence either F_2/F is the unique maximal such subgroup or $F_2=G$. In the former case, F_2/G' is the unique maximal θ-subgroup of G/G'; it follows readily that G/F_2 has prime order, say r, and that $G/F=gp\{\xi, F_2/F\}$, where ξ has order $r^k\geq r$ and $\xi^\theta=\xi^{\pm1}$.

Case 1: $Z(G/F)=1$. Evidently $F_2<G$. Let R/F be a Sylow r-subgroup of

G/F which contains ξ. If $|R/F|>r$, then $(R/F)\cap(F_2/F)>1$ and so $1<Z(R/F)\cap(F_2/F)\leq Z(G/F)$, contrary to assumption. Thus $k=1$ and F_2/F is an r'-group. Let T/F be a minimal normal subgroup of Γ/F contained in F_2/F. T/F is an elementary abelian q-group for some prime q. Since F_2/F is an r'-group, $q\neq r$. Since $F=F(G)$, $q\neq p$. It remains only to prove that $T=F_2$.

Suppose $T<F_2$. Set $K=gp\{\xi, T/F\}$. Since ξ has order r, $K/F<G/F$ and since $\xi^\theta=\xi^{\pm1}$, K/F is a θ-subgroup of G/F. By Lemma 3, K/F is nilpotent. In particular, ξ centralizes T/F. Then, since $T/F\leq Z(F_2/F)$, we have $T/F\leq Z(G/F)$, contrary to the assumption that $Z(G/F)=1$. Thus $T=F_2$ as required.

Case 2: $Z(G/F)>1$. We have to prove that this assumption leads to a contradiction. Choose $\zeta\in Z(G/F)$ such that $\zeta^\theta=\zeta^{\pm1}\neq1$. By Lemma 1, $\zeta=zF$, where $z^\theta=z^{\pm1}$. Since $gp\{\zeta\}\Gamma/F$, $C_F(z)$ is a normal subroup of Γ contained in F. By Lemma 2, $C_F(z)=1$. Then, by Corollary 3 to Lemma 2, $z^\theta=z^{-1}$.

Let $F_{-\theta}$ denote the group formed by the elements of F inverted by θ; then $F=F_\theta\times F_{-\theta}$. If $x\in F_{\pm\theta}\cap(F_{\pm\theta})^z$, then $x_\pm^{z^{-1}}=(x^{z^{-1}})^\theta=(x^\theta)^z=x^{\pm z}$, so that $x\in C_F(z^2)=1$; hence $F_{\pm\theta}\cap(F_{\pm\theta})^z=1$. It follows that $|F_{\pm\theta}|^2=|F_{\pm\theta}\times(F_{\pm\theta})^z|\leq|F|$. Therefore, since $F=F_\theta\times F_{-\theta}$, $|F_\theta|=|F_{-\theta}|=|F|^{1/2}$ and $F=F_\theta\times(F_\theta)^z$.

Let t be a p'-element of G_0. By Corollary 3 to Lemma 2, $F_\theta\leq C_F(t)$. Then $(F_\theta)^z\leq C_F(t)^z=C_F(t^z)=C_F(t)$, since $zF\in Z(G/F)$. Hence $F=F_\theta\times(F_\theta)^z\leq C_F(t)$ and so, by Lemma 2, $t=1$. Thus G_0 is a p-group. Since F_2/F is a p'-group, $(F_2/F)_\theta=1$. Thus θ inverts all elements of F_2/F and F_2/F is abelian. If G/F were a p'-group, the same argument would prove G/F abelian, contrary to (3.1). Hence $F_2<G$ and $r=p$. Also, since F_2/F is a p'-group, ξ has order p.

Set $Q=F_2/F$, $S=Z(G/F)$, $T=\Phi(S)$. Clearly, ξ does not centralize Q. On the other hand, the argument used in Case 1 shows that ξ centralizes every proper subgroup of Q which is normal in Γ/F. Thus S is the unique maximal such subgroup. It follows that Q is a q-group for some prime $q(\neq p)$. Also, since $Q/\Phi(Q)$ is a completely reducible (Γ/F_2)-module, it is in fact an irreducible module and thus $\Phi(Q)=S$. Since an automorphism of Q which leaves the elements of $Q/\Phi(Q)$ fixed has q-power order, ξ does not centralize Q/S.

Now, since Q is abelian, the mapping $xS\to x^qT(x\in Q)$ is a (Γ/F_2)-module epimorphism from Q/S to S/T. Since Q/S is irreducible and $S/T>1$, Q/S and S/T are isomorphic modules. Since ξ does not centralize Q/S, it does not centralize S/T. This is a contradiction because $S=Z(G/F)$. The proof is now complete.

4. We are now in a position to prove the Theorem. Let G be a counter-example of least order. Using the Corollary to Lemma 1, we see easily that G satisfies (3.1) and (3.2). Thus the conclusions of all the lemmas in Section 3 hold. In particular, $G/F=gp\{\xi, F_2/F\}$, where $\xi^r=1$ and $\xi^\theta=\xi^{\pm 1}$.

By Corollary 1 to Lemma 2, F provides a faithful, irreducible representation, say f, of Γ/F over the prime field $GF(p)$. Qua representation over the algebraic closure, k, of $GF(p)$, f splits into a direct sum of (absolutely) irreducible representations (cf Curtis and Reiner [1], § 70). Let V be a representation module (over k) corresponding to one of these irreducible parts. We use the customary (additive) module notation in V, writing scalars on the left, and group elements on the right, of vectors. If $x\in\Gamma/F$, V_x denotes the set of vectors fixed by x.

Since the absolutely irreducible parts of f are all algebraically conjugate (Curtis and Reiner l.c.), they all have the same kernel. Thus

(4.1) V is a faithful, irreducible (Γ/F)-module.

Using also Corollaries 2 and 3 of Lemma 2, we get

(4.2) $V_\theta>0$;

(4.3) if x is a p'-element of $(G/F)_\theta$, then $V_\theta\leq V_x$.

Case 1: V is a reducible (G/F)-module. Let W be an irreducible (G/F)-submodule of V, so that $0<W<V$. Then $W\theta$ is also a (G/F)-submodule of V; hence $W\cap W\theta$ and $W+W\theta$ are (Γ/F)-submodules and so $V=W\oplus W\theta$. Let x be a p'-element of $(G/F)_\theta$. If $w\in W$, then $w(1+\theta)\in V_\theta$ and so, by (4.3), $w(1+\theta)=w(1+\theta)x$. Hence $w(x-1)=-w\theta(x-1)=-w(x-1)\theta\in W\cap W\theta=0$, which shows that $w\in V_x$ and $w\theta\in V_x$. Thus $V_x=V$ and so, by (4.1), $x=1$. This proves that $(G/F)_\theta$ is a p-group. Hence $(G/F)_\theta\cap(F_2/F)=1$. But $(G/F)_\theta>1$, for otherwise G/F would be abelian, contrary to (3.1). It follows that $r=p$ and $\xi^\theta=\xi$.

Now $G>F_2$, so that hypothesis (ii) of the Theorem holds. Hence $V_\theta(\xi-1)^{p-1}=0$. By the argument used above (with $(\xi-1)^{p-1}$ in place of $(x-1)$), we deduce that $V(\xi-1)^{p-1}=0$. But this is contrary to Theorem B of Hall and Higman [4].

Case 2: V is an irreducible (G/F)-module. Let U be an irreducible (F_2/F)-submodule of V. Then U is a one-dimensional subspace, spanned by the vector u, say; also, $u\eta = \chi(\eta)u$ for $\eta \in F_2/F$, where χ is a character of F_2/F. Set $u_i = u\xi^{-i}$, $U_i = U\xi^{-i}$. Since $u_i\eta = u\eta^{\xi^i}\xi^{-i} = \chi(\eta^{\xi^i})u_i$, U_i is an (F_2/F)-submodule with corresponding character $\chi_i(\eta) = \chi(\eta^{\xi^i})$. Then $U_0 + \cdots + U_{r-1}$ is a (G/F)-submodule, so that $U_0 + \cdots + U_{r-1} = V$. Therefore, since V is a faithful module, χ is not the trivial character 1.

We prove next that $\chi_0, \ldots, \chi_{r-1}$ are all different. If not, then $\chi_i = \chi_j$ $(i \neq j)$ and this gives $\chi(\eta^{\xi^i - \xi^j}) = 1$ for all $\eta \in F_2/F$. Now the kernel of the endomorphism $\eta \to \eta^{\xi^i - \xi^j}$ of F_2/F is $C_{F_2/F}(\xi^{i-j}) = Z(G/F) = 1$. Therefore the image of this endomorphism is the whole of F_2/F and so our equations give $\chi = 1$, a contradiction. This proves the assertion. It follows that $U_0, \ldots, U_{r-1}$ are the only one-dimensional (F_2/F)-submodules of V and that $V = U_0 \oplus U_1 \oplus \cdots \oplus U_{r-1}$.

Since $(u_i\theta)\eta = \chi_i(\eta^\theta)(u_i\theta)$, θ permutes the U_i. Since the number, r, of the U_i is odd, θ leaves at least one U_i invariant. We may suppose $U\theta = U$.

Since U affords a 1-dimensional representation of $gp\{\theta, F_2/F\}$ and F_2/F is not contained in the kernel of this representation, the derived group of $gp\{\theta, F_2/F\}$ is a proper subgroup of F_2/F and so $(F_2/F)_\theta > 1$. Let $1 \neq \zeta \in (F_2/F)_\theta$. If $\xi^\theta = \xi$, then ξ, ζ are elements of coprime order in the nilpotent group $(G/F)_\theta$ and so $\xi\zeta = \zeta\xi$. But this implies that $\zeta \in Z(G/F)$, contrary to Lemma 4. Therefore $\xi^\theta = \xi^{-1}$, and consequently $U_i\theta = U_{-i}$ for every i.

The argument used in Case 1 shows that $u_i \in V_\zeta$ whenever $U_i\theta \neq U_i$. Hence

(4.4) $u_i \in V_\zeta$ for $1 \leq i \leq r-1$.

Since $u_i = u\xi^{-i}$, this gives

(4.5) $u\zeta^{\xi^i} = u$ for $1 \leq i \leq r-1$.

Now, since $\eta \to \eta^{\xi - 1}$ is an automorphism of F_2/F and $1 = \zeta^{\xi^r - 1} = \zeta^{(1+\xi+\cdots+\xi^{r-1})(\xi-1)}$, we have $\zeta^{1+\xi+\cdots+\xi^{r-1}} = 1$. With (4.5), this gives $u\zeta = u$. Hence, by (4.4), $V_\zeta = V$. Since this is contrary to (4.1), the proof is complete.

References

[1] C. W. Curtis and I. Reiner, Representation theory of finite groups and associative algebras. Interscience, New York, 1962.

[2] H. Fitting, Beiträge zur Theorie der Gruppen endlicher Ordnung. Jber. Deutsch.

Math.-Verein. **48** (1938), 77–141.

[3] W. Gaschütz, Über die Φ-Untergruppe endlicher Gruppen. Math. Z. **58** (1953), 160–170.

[4] P. Hall and Graham Higman, On the p-length of p-soluble groups and reduction theorems for Burnside's problem. Proc. London Math. Soc. (3) **6** (1956), 1–42.

[5] J. G. Thompson, Automorphisms of solvable groups. J. Algebra **1** (1964), 259–267.

[6] J. N. Ward, Involutory automorphisms of groups of odd order (to appear in J. Australian Math. Soc.).

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