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Cross varieties of groups

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One of the fundamental problems in the theory of varieties of groups is to decide whether the laws (identical relations) of each group admit a finite basis (in the sense that they are all consequences of a finite set of laws). Oates & Powell recently proved (1964) that the answer is affirmative in the case of finite groups. We present a considerably shortened proof of their result; with a little additional reasoning, this in fact yields a slight generalization of the Oates–Powell theorem.

1. Oates & Powell (1964) proved that a variety of groups is a Cross variety if and only if it can be generated by a finite group. This paper grew out of an analysis of their argument. The result here is the following:

Theorem. *Let e, m, c be positive integers, and let $\mathfrak{C}$ [or, in full, $\mathfrak{C}(e, m, c)$] be the class of groups G such that*

(i) *the exponent of G divides e,*
(ii) *every chief factor of G is of order at most m, and*
(iii) *every nilpotent factor of G is of class at most c.*

This class $\mathfrak{C}$ is a Cross variety: that is,

(*a*) *every finitely generated group in $\mathfrak{C}$ is finite,*
(*b*) *$\mathfrak{C}$ contains only finitely many (isomorphism classes of) critical groups, and*
(*c*) *$\mathfrak{C}$ is a variety whose laws are finitely based.*

This yields a generalization of theorem 3 of Oates & Powell (1964).

Corollary. *A variety generated by a class $\mathfrak{X}$ of groups is a Cross variety if and only if $\mathfrak{X}$ is contained in some class $\mathfrak{C}(e, m, c)$.*

To derive the corollary, note that every Cross variety can be generated by a finite group (from theorem 1 in Higman 1960) and each finite group is contained in some class $\mathfrak{C}$ (e, m, c), while on the other hand every subvariety of a Cross variety is a Cross variety (theorem 2 in Higman 1960).

The proof of the theorem is so arranged that its first three parts, §§2, 3, 4, constitute a direct proof of the Oates–Powell result (which appears here as (4·5)). Part (*b*) of the theorem is implicit in the proof of theorem 4 in Oates & Powell (1964), and the proof given in §3 is a simplified version of what can be found there. The main new idea is the use of certain laws which arose in another investigation of ours and whose relevant properties are described in (4·3) and (4·4). After extracting (*b*) from Oates & Powell (1964), one of us (L. G. K.) recognized that these laws make it possible to complete the proof of the Oates–Powell result by the argument in §§2 and 4.

For terminology, the reader is referred to Oates & Powell (1964). There is one exception: the use of 'law' for 'identical relation'.

We thank Drs Oates and Powell for making a pre-publication manuscript available, Professor Hanna Neumann for presenting it in a course of lectures, and Professor B. H. Neumann for stimulating conversations.

2. The following simple remark will be used repeatedly:

REMARK 2·1. If H/K is a chief factor of a group G and C is the centralizer of H/K in G, then G/C is faithfully represented by automorphisms of H/K: hence, if $|H/K| \leqslant m$, then—to be very generous—$|G/C| < m!$. Conversely, if $|G/C| \leqslant n$, then either $|H/K| \leqslant n$ or H/K is an elementary Abelian group on at most n generators; thus, if G is a group of exponent dividing e, then $|G/C| \leqslant n$ implies that $|H/K| \leqslant e^n$.

Proof of (a).

Let G be a finitely generated group in $\mathfrak{C}(e, m, c)$; let F be the intersection of the centralizers of the chief factors of G, and K the intersection of all the normal subgroups of finite index in G. By (ii) and (2·1), $K \leqslant F$. Being a finitely generated group, G has only finitely many normal subgroups of index at most $m!$ (see, for example, lemma 3·4 of Neumann 1955); hence, by (2·1), $|G/F|$ is finite. Thus Schreier's theorem (see, for example, M. Hall 1959, p. 97, corollary 7·2·1) implies that F is finitely generated. Let N be a normal subgroup of finite index in G. Consider a chief series of the finite group $G/F \cap N$ through $F/F \cap N$: all the chief factors which lie in $F/F \cap N$ are central in $F/F \cap N$, so $F/F \cap N$ is nilpotent. Now $F/F \cap N$ can be generated by at most as many elements as F; by (iii), it has class at most c; by (i), its exponent divides e: hence there is a bound on the order of $F/F \cap N$ which is independent of N. Thus there is a bound on the orders of all finite factor groups of G; hence, by the argument above, G has only finitely many normal subgroups of finite index: it follows that $|G/K|$ is finite. By Schreier's theorem, K is finitely generated. Thus, if K were nontrivial, G would have a chief factor of the form K/L; but then, by the definition of K, $|G/L|$, and hence also $|K/L|$, would be infinite, contrary to (ii). This proves that $K = 1$ and so G is finite, as required.

3. *Proof of* (b)

The key to this is lemma 2·4·2 of Oates & Powell (1964):

LEMMA 3·1. If a group G has a set of normal subgroups $M_1, \ldots, M_s$ and a subgroup L such that

$$G = LM_1, \ldots, M_s; \tag{3·11}$$

G is not generated by L together with any proper subset of $\{M_1, \ldots, M_s\}$; (3·12)

$[M_{\pi(1)}, \ldots, M_{\pi(s)}] = 1$ for every permutation π of the integers $1, \ldots, s$; (3·13)

then G is not critical.

It is necessary to apply this lemma in two different situations. The first application needs hardly any preparation; we begin the proof with this.

Let G be a critical group in $\mathfrak{C}(e, m, c)$, F the Fitting subgroup and Φ the Frattini subgroup of G. Then

$$|F/\Phi| \leqslant m^c. \tag{3·2}$$

Proof. By theorems 1 and 7 of Gaschütz (1953), F/Φ is a direct product of minimal normal subgroups $M_1/\Phi, \ldots, M_s/\Phi$ of G/Φ; moreover, F/Φ has a complement L/Φ in G/Φ. These subgroups $L, M_1, \ldots, M_s$ obviously satisfy (3·11) and (3·12).

Each M_i/Φ is a chief factor of G and so has order at most m. If (3·2) were false, s would have to be greater than c; by (iii), F is of class at most c; hence in this case $M_1, \ldots, M_s$ would satisfy (3·13) as well, contrary to the criticality of G.

Note that the centralizer C of F/Φ in G is the intersection of the centralizers of the chief factors M_i/Φ; by (ii) and (2·1), each of these has index at most $m!$; as we have seen, there are at most c such centralizers involved: hence it follows that

$$|G/C| \leqslant (m!)^c. \tag{3·3}$$

The plan for the rest of the proof of (b) is the following. It will be shown that $|G/\Phi|$ can be bounded in terms of e, m, c; say, $|G/\Phi| \leqslant b$. Then G can be generated by b elements, and Φ is a subgroup of index at most b in G: hence, by Schreier's theorem, Φ can be generated by b' elements where b' is finite and depends only on b, i.e. on e, m, c. Now Φ is nilpotent of class at most c and has exponent dividing e: thus $|\Phi|$ can be bounded in terms of b', c, e. Consequently, $|G|$ can be bounded in terms of e, m, c, and hence (up to isomorphism) there can only be finitely many critical groups in $\mathfrak{C}$.

In view of (3·2), it suffices therefore to show that $|G/F|$ can be bounded in terms of e, m, c. Now it is well known (see, for example, Hall 1940, p. 210) that F is precisely the intersection of the centralizers of the chief factors of G; in particular, $F \leqslant C$. Thus (3·3) further reduces the problem to that of bounding $|C/F|$. Let $F = \bigcap (C \cap C_\tau : 1 \leqslant \tau \leqslant t)$ with each C_τ the centralizer of some chief factor of G and t as small as possible. Put $C^* = \bigcap (C_\tau : 1 \leqslant \tau \leqslant t)$; then $F = C \cap C^*$ and so $|C/F| = |CC^*/C^*| \leqslant |G/C^*| \leqslant (m!)^t$, by (ii) and 2·1: hence

$$\textit{it suffices to give a bound for } t. \tag{3·4}$$

For this, one has to consider only the case $t > 1$. By the minimality of t, none of the subgroups D_τ defined by $D_\tau = \bigcap (C \cap C_\sigma : 1 \leqslant \sigma \leqslant t, \sigma \neq \tau)$ is equal to F. Moreover the product $\Pi(D_\tau/F : 1 \leqslant \tau \leqslant t)$ is direct: for, if $\Pi(d_\tau : 1 \leqslant \tau \leqslant t) \in F$ with $d_\tau \in D_\tau$ for each τ, then

$$(d_\tau F)^{-1} = \Pi(d_\sigma F : 1 \leqslant \sigma \leqslant t, \sigma \neq \tau) \in D_\tau/F \cap C_\tau/F = F/F \quad \text{for} \quad 1 \leqslant \tau \leqslant t$$

(note that in these products the order of the factors is immaterial since the normal subgroups D_τ/F are pairwise disjoint). Let N_τ/F be arbitrary minimal normal subgroups of G/F in the D_τ/F, one for each τ.

By theorem 10 of Gaschütz (1953), each N_τ/Φ is non-nilpotent. On the other hand, as $N_\tau \leqslant C$, F/Φ is central in N_τ/Φ: it follows that every N_τ/F is non-nilpotent. As the product $\Pi(D_\tau/F : 1 \leqslant \tau \leqslant t)$ is direct, so is $\Pi(N_\tau/F : 1 \leqslant \tau \leqslant t)$. Now lemma 2·2·9 of Oates & Powell (1964) gives that G/F has a subgroup T/F such that G/F, T/F, $N_1/F, \ldots, N_t/F$ (in place of G, L, $M_1, \ldots, M_s$) satisfy conditions (3·11) and (3·12); hence $G, T, N_1, \ldots, N_t$ also satisfy these conditions. The remaining steps will show that if $t \geqslant 2 + m!ce$, then the N_t also satisfy (3·13), so that a second application of (3·1) gives a contradiction to the criticality of G. Thus it will follow that $t < 2 + m!\,ce$ which, in view of (3·4), completes the proof of (b).

Let $F = F_0 > \ldots > F_k = 1$ be the series obtained by refining the lower central series of F with terms corresponding to the lower Frattini series of its factors:

then each F_κ is characteristic in F and hence normal in G, while all the factors $F_{\kappa-1}/F_\kappa$ are elementary Abelian and central in F, for $\kappa = 1, \ldots, k$. (The case $F = 1$ can be disregarded, for then G/F has only one minimal normal subgroup and so $t \leqslant 1$.) As F has class at most c and exponent dividing e, $k < ce$ is a very generous estimate.

Since G has only one minimal normal subgroup and F is nilpotent, F has only one Sylow subgroup: so F is a p-group for some prime divisor p of e. The key to the final step is the following proposition.

If $1 \leqslant \kappa \leqslant k$ and if $b_1F, \ldots, b_{m!}F$ are elements of orders prime to p from distinct N_τ/F, then

$$[F_{\kappa-1}, b_1, \ldots, b_{m!}] \leqslant F_\kappa. \tag{3·5}$$

To prove this assertion, consider the section $F_{\kappa-1} = X_0 > \ldots > X_r = F_\kappa$ of a chief series of G through $F_{\kappa-1}$and F_κ; further, let B be the subgroup generated by F and the elements $b_1, \ldots, b_{m!}$. As F centralizes $F_{\kappa-1}/F_\kappa$ and B/F is a direct product of $m!$ (cyclic) groups of orders prime to p, Maschke's theorem implies that each $X_{\rho-1}/F_\kappa$ is the direct product of the corresponding X_ρ/F_κ and a suitable normal subgroup Y_ρ/F_κ of B/F_κ, for $\rho = 1, \ldots, r$. Now $F_{\kappa-1}/F_\kappa$ is the direct product of the Y_ρ/F_κ, and it suffices to show that $[Y_\rho, b_1, \ldots, b_{m!}] \leqslant F_\kappa$ or, equivalently, that

$$[X_{\rho-1}, b_1, \ldots, b_{m!}] \leqslant X_\rho$$

for each ρ. In turn, this will follow if one can show that at least one of the elements $b_1, \ldots, b_{m!}$ belongs to the centralizer K_ρ of the chief factor $X_{\rho-1}/X_\rho$ of G. Consider $K_\rho/F \cap \Pi(N_\tau/F: 1 \leqslant \tau \leqslant t)$: this is a normal subgroup of G/F contained in the direct product of the non-Abelian minimal normal subgroups N_τ/F. As is well known and is in any case easy to prove†), this implies that $K_\rho/F \cap \Pi(N_\tau/F: 1 \leqslant \tau \leqslant t)$ is in fact the product of some of the N_τ/F. Thus the direct product of those N_τ/F which lie outside K_ρ/F avoids K_ρ/F; by (ii) and (2·1), $|G/K_\rho| < m!$ and so it follows that K_ρ contains all but at most $m!-1$ of the N_τ. Hence of the $m!$ elements $b_1, \ldots, b_{m!}$, which were taken from $m!$ distinct N_τ, at least one must lie in K_ρ. This completes the proof of (3·5).

To complete the proof of (*b*), let $b_\tau F$ be elements of orders prime to p, one from each N_τ/F and suppose that $t \geqslant 2+m!\,ce$. As N_1/F and N_2/F commute

$$[b_1, b_2] \in F = F_0.$$

By (3·5) it follows that $[b_1, \ldots, b_{2+m!}] \in F_1$; $(k-1)$ further repetitions of this argument give that $[b_1, \ldots, b_{2+km!}] \in F_k = 1$; and then $t \geqslant 2+m!\,ce > 2+k\,m!$ gives that $[b_1, \ldots, b_t] = 1$. Since the N_τ/F are non-Abelian chief factors of G, they are direct

† Let H be a normal subgroup of an arbitrary group K, and suppose that H is contained in the direct product of certain non-Abelian minimal normal subgroups $H_1, \ldots, H_n$ of K. In proving that H is in fact the product of some of the H_i, it can be assumed without loss of generality that H is not contained in the product of less than n of the H_i. Then H has elements h_j of the form $h_j = h_{1j} \ldots h_{nj}$ with $h_{ij} \in H_i$ and $h_{ii} \neq 1$, for $i, j = 1, \ldots, n$. Since the H_i have trivial centres, they have elements h_i' such that $[h_{ii}, h_i'] \neq 1$. Now

$$1 \neq [h_{ii}, h_i'] = [h_i, h_i'] \in H \cap H_i \leqslant H_i;$$

as $H \cap H_i$ is normal and H_i is minimal normal in K, it follows that $H_i = H \cap H_i \leqslant H$ for each i. This proves that $H = H_1 \ldots H_n$.

powers of non-Abelian simple groups, and so each N_τ is generated by those elements b_τ (of N_τ) whose cosets mod F have orders prime to p. By a routine argument (see, for example, lemma 2·4 of P. Hall (1957), the fact that $[b_1, \ldots, b_t] = 1$ holds for every choice of the b_τ from these (characteristic) generating sets of the N_τ implies that $[N_1, \ldots, N_t] = 1$. This is true regardless of the order in which the N_τ were listed: hence the N_τ satisfy (3·13), and the proof of (b) is complete.

4. The next major aim is to prove the following proposition.

If G is a finite group, then there is a integer k such that the variety defined by those laws of G which involve at most k variables is contained in some class $\mathfrak{C}$. (4·1)

Three simple lemmas will be required.

The class of finite groups within a class $\mathfrak{C}$ is closed with respect to taking subgroups, factor groups, and finite direct products. (4·2)

The proof of this is straightforward modulo the observation that a finite group satisfies (iii) if and only if its Sylow sugbroups are all of class at most c.

The other two lemmas concern the following words:

$$u_3 = [(x_1^{-1}x_2)^{x_{12}}, \quad (x_1^{-1}x_3)^{x_{13}}, \quad (x_2^{-1}x_3)^{x_{23}}]$$

and
$$u_n = [u_{n-1}, (x_1^{-1}x_n)^{x_{1n}}, \ldots, (x_{n-1}^{-1}x_n)^{x_{n-1,n}}] \quad \text{if} \quad n > 3.$$

If G is a finite group of order less than n and $n \geqslant 3$, then $u_n = 1$ is a law in G. (4·3)

Proof. Since G does not contain n distinct elements, at least one of the $g_i^{-1}g_j$ is the identity for every set $g_1, \ldots, g_n, g_{12}, \ldots, g_{n-1,n}$ of elements of G that one substitutes for the variables of u_n, and hence u_n has no nontrivial value in G.

If the centralizer C of a chief factor H/K of a group G has index at least n in G, then $u_n = 1$ is not a law in G. (4·4)

Proof. Put $g_1 = 1$; let $g_2, \ldots, g_n$ be representatives of distinct cosets of C with $g_2 \in H \backslash K$ and $g_3, \ldots, g_n \notin C$. Consider the normal closure N_{13} of $g_1^{-1}g_3$ and K in G. The centralizer of N_{13}/K in G/K is a normal subgroup which does not contain $g_1^{-1}g_2K$: for otherwise it would also contain the normal closure H/K of $g_1^{-1}g_2K$, and then $[H/K, g_1^{-1}g_3K] = 1$, i.e. $[H, g_1^{-1}g_3] \subseteq K$ and so $g_1^{-1}g_3 \in C$ would follow, contrary to the initial assumptions. Thus $[g_1^{-1}g_2, N_{13}] \nleq K$, so that there is a conjugate $(g_1^{-1}g_3)^{g_{13}}$ of $g_1^{-1}g_3$ for which $[g_1^{-1}g_2, (g_1^{-1}g_3)^{g_{13}}] \in H \backslash K$. Repetition of the argument with this commutator in place of $g_1^{-1}g_2$ and $g_2^{-1}g_3$ in place of $g_1^{-1}g_3$ gives that, for some g_{23},

$$[g_1^{-1}g_2, (g_1^{-1}g_3)^{g_{13}}, (g_2^{-1}g_3)^{g_{23}}] \in H \backslash K.$$

This commutator now is a value of u_3 in G. If $n > 3$, a further $\frac{1}{2}n(n-1) - 3$ repetitions of the twice used argument gives finally that u_n has a value in G which lies in $H \backslash K$ and is therefore nontrivial.

Proof of (4·1). Consider an arbitrary nontrivial, finite group G. Let n denote the order of G, e the exponent of G, and c the maximum of the classes of Sylow subgroups of G; choose k to be $\frac{1}{2}(n+1)(n+2)$, the number of variables in u_{n+1}; and let $\mathfrak{U}$ be the variety defined by the (at most) k-variable laws of G. It can now be shown that $\mathfrak{U} \subseteq \mathfrak{C} = \mathfrak{C}(e, e^n, c)$. Clearly, $G \in \mathfrak{C}$. Let H be an arbitrary group in $\mathfrak{U}$. With G, H satisfies the law $x^e = 1$, and so has exponent dividing e. By (4·3), $u_{n+1} = 1$ is a law in $\mathfrak{U}$; by (4·4), it follows that every chief factor of H has centralizer of index at most n in H; by (2·1), this implies that every chief factor of H is of order at most e^n. In order to prove that $H \in \mathfrak{C}$, it remains to show that every nilpotent factor K of H which can be generated by $c+1$ elements is of class at most c. By the definition of $\mathfrak{U}$ (as $k > c+1$), these factors K all belong to the variety generated by G; according to lemma 4·3 of Higman (1959), each K is therefore a factor group of a subgroup of a finite direct power of G. In view of $G \in \mathfrak{C}$ and (4·2), this means that each K belongs to $\mathfrak{C}$ and therefore has class at most c. Thus indeed $H \in \mathfrak{C}$ and so it has been proved that $\mathfrak{U} \subseteq \mathfrak{C}$.

According to a result of B. H. Neumann (read theorem 14·2 of Neumann (1937) in conjunction with Schreier's theorem) the laws of $\mathfrak{U}$ are finitely based. Thus (4.1) and (*a*), (*b*) give that $\mathfrak{U}$ is a Cross variety. As the variety generated by G is contained in $\mathfrak{U}$, and as subvarieties of Cross varieties are Cross varieties, the main result of Oates & Powell (1964) has been reached:

If G is a finite group, then the variety generated by G is a Cross variety. (4·5)

5. Only two steps of the proof of the theorem remain.

If G is a finite group in $\mathfrak{C} = \mathfrak{C}(e, m, c)$, then the whole variety $\mathfrak{V}$ generated by G in contained in $\mathfrak{C}$. (5·1)

Proof. The case $G = 1$ can be ignored; let n be the order of G, $n > 1$, and let H be an arbitrary group in $\mathfrak{V}$. According to the proof of (4·1), $H \in \mathfrak{C}(e, e^n, c)$; so every chief factor A/B of H is finite and—in order to prove that $H \in \mathfrak{C}(e, m, c)$—the only thing to verify is that $|A/B| \leqslant m$. To each of the finitely many non-trivial proper subgroups of A/B, choose an element from H/B which does not normalize it; consider the subgroup K/B of H/B generated by these elements together with A/B. Then K/B is a finitely generated group in $\mathfrak{V}$, and A/B is a chief factor of K/B. By lemma 4·3 of Higman (1959), K/B is a factor group of a subgroup of a finite direct power of G; hence, according to (4·2), $K/B \in \mathfrak{C}(e, m, c)$: thus $|A/B| \leqslant m$, as required.

For the final step, let $\mathfrak{C} = \mathfrak{C}(e, m, c)$ be fixed; according to (*b*), $\mathfrak{C}$ contains only finitely many isomorphism classes of critical groups; let G be the direct product of a complete set of representatives of these isomorphism classes, and let $\mathfrak{V}$ be the variety generated by G. In view of (4·5) and (5·1), the theorem will follow from the following assertion:

$$\mathfrak{C} \subseteq \mathfrak{V}. \tag{5·2}$$

Proof. It suffices to show that if $H \in \mathfrak{C}$ then every finitely generated subgroup K of H lies in $\mathfrak{V}$. Suppose this is not so: then the relevant verbal subgroup $V(K)$ of K is nontrivial. As $K/V(K)$ is a finitely generated group in $\mathfrak{V}$ and hence, by (5·1), in $\mathfrak{C}$, (*a*) implies that $K/V(K)$ is finite: thus it follows from Schreier's theorem that $V(K)$ is finitely generated. Therefore there exists a normal subgroup N of H which is maximal with respect to not containing $V(K)$. The normal subgroups of H which properly contain N all contain $V(K)$, and so their intersection M is greater than N. Now M/N is a chief factor of H, and so $|M/N| \leqslant m$. Put

$$L = V(K) \cap N,$$

note that $V(K) > L$, and consider K/L. This is a factor of H, and so has exponent dividing e and all its nilpotent factors are of class at most c. Moreover,

$$V(K)/L \cong V(K)N/N \leqslant M/N, \quad \text{so} \quad |V(K)/L| \leqslant m,$$

while $K/V(K) \in \mathfrak{V} \subseteq \mathfrak{C}$: so K/L is finite, and (by an application of the Jordan–Hölder theorem) the chief factors of K/L are all of order at most m: thus $K/L \in \mathfrak{C}$. Let $\mathfrak{W}$ be the variety generated by K/L. By theorem 1 in Higman (1960), $\mathfrak{W}$ is generated by its critical groups; in view of (5·1), $\mathfrak{W} \subseteq \mathfrak{C}$, and so every critical group of $\mathfrak{W}$ is isomorphic to a direct factor of G: thus $K/L \in \mathfrak{W} \subseteq \mathfrak{V}$. This contradicts $V(K) > L$. Thus $V(K) = 1$, $K \in \mathfrak{V}$, $H \in \mathfrak{V}$, and $\mathfrak{C} \subseteq \mathfrak{V}$, as required. The proof of (5·2) and with it the proof of the theorem, is now complete.

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