Gilbert Baumslag, L. G. Kovács, and B. H. Neumann

On products of normal subgroups

SZEGED, 1965
On products of normal subgroups

By GILBERT BAUMSLAG, L. G. KOVÁCS and B. H. NEUMANN in Canberra (Australia)

The purpose of this note is to demonstrate that the product of two "good" normal subgroups of a group need not be "good": we do this for the cases when "good" is interpreted, in turn, as locally soluble, locally $SI^*$, $SI$, locally $SN^*$, locally residually nilpotent, and residually nilpotent. The first four interpretations answer questions recorded in section 2. 4 of PLOTKIN'S survey [3]1; the fifth answers question 13. 1. 3 of [3]; and the last contradicts an assertion of SESEKIN ([4], part of Corollary 1 to Lemma 8) quoted in [3] (in the paragraph immediately preceding 13. 1. 3). In fact, we construct two examples which show even more:

Theorem 1. There exists a finitely generated group which is the product of two locally soluble normal subgroups but is neither an $SI$-group nor a radical group (in the sense of PLOTKIN [3]; note that, according to § 15 of [3], it follows that the group is not an $SN^*$-group).

Theorem 2. There exists a finitely generated (torsion-free, metabelian) group which is the product of two residually nilpotent normal subgroups but is not residually nilpotent.

The proofs depend on the following:

Lemma. To each group $H$ which is a split extension of a group $G$ by an abelian group $A$, it is possible to construct a group $H^*$ which contains a subgroup isomorphic to $H$ and is the product of two normal subgroups isomorphic to the restricted (standard) wreath product $G$ wr $A$. Moreover, if $H$ is finitely generated, then so is the corresponding $H^*$.

A similar construction has been used independently by HALL (unpublished) for dealing with some of PLOTKIN'S questions which are answered by Theorem 1. A result similar to Theorem 2 can also be deduced from a theorem of HALL and HARTLEY (to appear) according to which every group is embeddable in a suitable product of two normal free subgroups.

Proof of the Lemma. Let $H$ be a split extension of a group $G$ by an abelian group $A$; it can be assumed that $G$ is a normal subgroup of $H$, $A \cap H = 1$, and there

---

1) The remaining question of this type in 2. 4 of [3], namely the question relating to the class $SN$, has a positive answer: it is straightforward to see that even all extensions of $SN$-groups by $SN$-groups are $SN$-groups.

*) The first author acknowledges support from the National Science Foundation of the U.S.A.
is a monomorphism $\beta: A \to H$ for which $G \cap A\beta = 1$ and $G(A\beta) = H$. Consider the unrestricted (standard) wreath product $P$ of $H$ and $A$, and write its base group $H^A$ as the group of functions from $A$ to $H$ with valuewise multiplication. Call $K$ that subgroup of $H^A$ which consists of those functions whose values are all in $G$ and are in fact equal to 1 at all but finitely many elements of $A$. This $K$ is normal in $P$, and $KA \cong G$ wr $A$. For each element $a$ in $A$, let $a\delta$ be the constant function on $A$ with value $a\beta$; then $\delta: A \to H^A$ is a monomorphism; moreover, $A\delta$ and $A$ generate an abelian subgroup in $P$. We need next the element $f$ of $H^A$ defined by

$$f(a)=a\beta \quad \text{for every } a \in A.$$  

Straightforward calculation shows that

$$f^{-1}af=a\delta \cdot a$$

so that $A^f \cong (A\delta)A$, and therefore $A$ and $A^f$ commute elementwise. Consequently, $KA$ and $KA^f$ normalize each other, and so their product $H^*$ is a subgroup of $P$. As $KA^f = (KA)^f$, we have that $H^*$ is the product of two normal subgroups isomorphic to $G$ wr $A$. To each element $g$ of $G$, let the element $g\gamma$ of $H^*$ be defined by

$$(g\gamma)(1) = g, \quad (g\gamma)(a) = 1 \quad \text{whenever } 1 \neq a \in A.$$ 

Then $\gamma: G \to H^*$ is a monomorphism; moreover, $G\gamma \cong K$. Each element of $H$ is uniquely a product $g(a\beta)$ with $g \in G$, $a \in A$; the mapping $\alpha: g(a\beta) \to (g\gamma)(a\delta)$ is therefore well defined; in fact, it is a monomorphism of $H$ into $H^A$. As $G\gamma \cong K$, and as (*) shows that $A\delta \cong AA^f$, it follows that $H\alpha \cong H^*$: so $H$ has a subgroup isomorphic to $H$. Finally, suppose that $H$ is finitely generated, and note that in this case so is $A$. It is easily seen that $G\gamma$ and $A$ generate $KA$; hence $H^*$ is generated by $G\gamma$, $A\delta$, and $A$. The subgroup generated by $G\gamma$ and $A\delta$ is precisely $H\alpha$. Thus we conclude that $H^*$ is generated by the finitely generated subgroups $H\alpha$ and $A$, and so $H^*$ itself is finitely generated.

**Proof of Theorem 1.** We use the terminology and results of **HALL** [1]. Let $G$ be the wreath power $\mathrm{Wr} C^2$ of an infinite cyclic group $C$ corresponding to the naturally ordered set $Z$ of rational integers, and let $A$ be the group of all order-preserving permutations of $Z$. Then $A$ is again an infinite cyclic group, and there is a natural split extension $H$ of $G$ by $A$. Clearly, this $H$ is finitely generated. According to **Theorem D** of **HALL** [1], $G'$ is a minimal normal subgroup of $H$. It is easy to see that $G'$ is not locally nilpotent; consequently, no group containing $H$ can be an $SI$-group or a radical group. On the other hand, $G$ is locally soluble and therefore so is $G$ wr $A$. The corresponding group $H^*$ of the Lemma provides the example required for the theorem.

**Proof of Theorem 2.** Let $G$ be the group defined on the generators $g_1, g_2, \ldots$ by the relations $g_i = g_{i+1}^2$, $i = 1, 2, \ldots$; then $G$ is an abelian group of rank 1. Let $A$ be the group of automorphisms of $G$ generated by the automorphism $\alpha: g \to g^2$, and let $H$ be the natural split extension of $G$ by $A$, with $a$ an element from that coset of $G$ in $H$ which corresponds to $\alpha$. Then $g_1$ and $a$ generate $H$; moreover, the relation

$$[g, a] = g \quad \text{for every } g \in G$$
On products of normal subgroups

shows that $G$ lies in every term of the lower central series of $H$: consequently, $H$ is not residually nilpotent, and so no group containing $H$ can be residually nilpotent. On the other hand, according to Lemma 14 of HALL [2], $G \wr A$ is residually nilpotent. Thus the corresponding group $H^*$ of the Lemma provides the required example.

References


THE AUSTRALIAN NATIONAL UNIVERSITY

(Received August 12, 1964)