ON A PAPER OF LADISLAV PROCHÁZKA

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The purpose of this note is to present two theorems and an example.

An abelian group is said to *split* if it is the direct sum of its maximal torsion subgroup and a torsion free subgroup.

Theorem A. *Let G be an abelian group, n a positive integer, and H a subgroup of G such that $nG \subseteq H$.*

A1. *If H splits and its torsion free component is a direct sum of groups of finite rank, then G also splits.*

A2. *If G splits and its torsion free component is a direct sum of groups of finite rank, then H also splits.*

Theorem B. *Let G be a torsion free abelian group, n a positive integer, and H a subgroup of G such that $nG \subseteq H$.*

B1. *If H is a direct sum of groups of rank* 1 *whose types are inversely well-ordered in the natural partial order of types, then $H \cong G$.*

B2. *If G is a direct sum of groups of rank* 1 *whose types are inversely well-ordered in the natural partial order of types, then $G \cong H$.*

Example. There is a direct sum U of torsion free abelian groups of rank 1 whose types are well-ordered in the natural partial order of types, which has a subgroup V such that $2U \subseteq V$ but V is not a direct sum of groups of rank 1.

Theorem A generalizes the first halves of Theorems 1 and 2 of L. PROCHÁZKA's paper [3], where in a similar situation the conclusion of splitting was reached under more stringent conditions on the torsion free components of H and G respectively. The other halves of these theorems stated that the torsion free components of H and G are isomorphic. The example shows (with $G = V$, $H = 2U$ for Theorem 1 and $G = U$, $H = V$ for Theorem 2) that for this, even in the torsion free case, his conditions have to be strengthened as is done in Theorem B.

The main tool in the proof of Theorem A is Theorem 5 of L. Procházka [3]. This is the special case of Theorem A which arises when H or G, respectively for A1 or A2, is taken to have finite torsion free rank. Several steps from Procházka's paper are incorporated in the proofs here, though item-by-item acknowledgements are omitted for the sake of fluency. The terminology and notation conforms to that in L. Fuchs's book [2].

Proof of A1. Let $H = P + \sum(A_\alpha : \alpha < \sigma)$ where P is a torsion group, the A_α are torsion free groups of finite rank, σ is an ordinal, and α runs through all the ordinals that precede σ. If T is the maximal torsion subgroup of G and β is an ordinal such that $\beta \leqq \sigma$, then clearly $T \cap \sum(A_\alpha : \alpha < \beta) = 0$. Moreover, if H_β is the subgroup generated by T and the A_α with $\alpha < \beta$, one has that:

(1) if $\beta \leqq \gamma \leqq \sigma$, then $H_\beta \subseteq H_\gamma$;

(2) if $\lambda \leqq \sigma$ and λ is a limit ordinal, then $H_\lambda = \bigcup (H_\beta : \beta < \lambda)$;

(3) if $\beta \leqq \gamma \leqq \sigma$, then $H_\gamma = H_\beta + \sum(A_\alpha : \beta \leqq \alpha < \gamma)$; and

(4) $nG \subseteq H \subseteq H_\sigma \subseteq G$.

The factor group G/T is torsion free, and so one can speak of the unique pure closure G_β/T of H_β/T in G/T, for each relevant β.

It is easy to see that

$$(5) \qquad G_\beta \cap H_\gamma = H_\beta\,, \quad \text{for each} \quad \beta \leqq \gamma \leqq \sigma\,.$$

For this, it suffices to show that $G_\beta \cap H_\gamma \subseteq H_\beta$, as the converse inclusion holds by definition. Let $0 \neq g \in G_\beta \cap H_\gamma$, and let $g = g' + g''$ according to the direct decomposition (3) of H_γ. As G_β/T is the pure closure of H_β/T, some non-zero multiple $mg + T$ of $g + T$ is contained in H_β/T, and then $mg = mg' + mg'' \in H_\beta$. This implies that $mg'' = 0$, and hence, $\sum(A_\alpha : \beta \leqq \alpha < \gamma)$ being torsion free, it follows that $g'' = 0$, $g \in H_\beta$.

The next thing to observe is that

(6) if $\beta \leqq \gamma \leqq \sigma$, then $G_\beta \subseteq G_\gamma$.

This clearly follows from (1) and from the definition of the G_β. Similarly,

(7) if $\lambda \leqq \sigma$ and λ is a limit ordinal, then $G_\lambda = \bigcup (G_\beta : \beta < \lambda)$.

To prove this, one shows that $G_\lambda/T = \bigcup (G_\beta/T : \beta < \lambda)$. According to (6), the union U on the right hand side comes from an ascending chain of pure subgroups; as such, it is itself pure in G/T; it contains each H_β/T for $\beta < \lambda$, and so it contains their union which is, according to (2), just H_λ/T; hence it also contains the pure closure G_λ/T of H_λ/T in G/T. For the converse inclusion, note that every element of U is in some G_β/T, that is in the pure closure of some H_β/T, with $\beta < \lambda$; thus it is a fortiori contained in the pure closure G_λ/T of H_λ/T.

The main step in the proof is the inductive selection of subgroups B_β, one for each β with $\beta \leqq \sigma$, such that $G_\beta = T + B_\beta$ and $B_\beta \subseteq B_\gamma$ whenever $\beta \leqq \gamma \leqq \sigma$. The initial step is a trivial one: as $H_0 = T$, also $G_0 = T$, and so one can take $B_0 = 0$. Suppose now that $0 < \gamma \leqq \sigma$, that the B_β have already been selected for each β which precedes γ, and that they all satisfy the above requirements.

If $\gamma - 1$ exists, choose B_γ as follows. Let $G' = G_\gamma/B_{\gamma-1}$. As $A_{\gamma-1} \cap (B_{\gamma-1} + T) = A_{\gamma-1} \cap G_{\gamma-1} = A_{\gamma-1} \cap H_\sigma \cap G_{\gamma-1} = A_{\gamma-1} \cap H_{\gamma-1} = 0$ [by (4), (5), and (3)], the subgroup H^* generated by $A_{\gamma-1}$, $B_{\gamma-1}$, and T is their direct sum $A_{\gamma-1} + B_{\gamma-1} + T$. Let $H' = H^*/B_{\gamma-1}$; then $H' \subseteq G'$. It is seen that H' splits, and that its torsion free component is of finite rank. Moreover, as $nG_\gamma \subseteq nG \subseteq H \subseteq H_\sigma$, also $nG_\gamma \subseteq G_\gamma \cap H_\sigma = H_\gamma$, according to (5). Since $H_\gamma = H_{\gamma-1} + A_{\gamma-1} \subseteq G_{\gamma-1} + A_{\gamma-1} = H^*$, it follows that $nG' \subseteq H'$. Hence, by Theorem 5 of Procházka [3], G' splits. The maximal torsion subgroup of G' is precisely $G_{\gamma-1}/B_{\gamma-1}$; for $G'/(G_{\gamma-1}/B_{\gamma-1})$ is isomorphic to $G_\gamma/G_{\gamma-1}$ and is therefore torsion free (in view of the purity of $G_{\gamma-1}/T$ in G/T), while $G_{\gamma-1}/B_{\gamma-1} \cong T$. Thus the splitting of G' means that $G' = B_\gamma/B_{\gamma-1} + G_{\gamma-1}/B_{\gamma-1}$ for some subgroup B_γ. Now B_γ and $G_{\gamma-1}$ generate G_γ; as $B_{\gamma-1} \subseteq B_\gamma$ and $G_{\gamma-1} = B_{\gamma-1} + T$, it follows that B_γ and T also generate G_γ; moreover, $B_\gamma \cap T \subseteq B_\gamma \cap G_{\gamma-1} \cap T = B_{\gamma-1} \cap T = 0$. Thus $G_\gamma = B_\gamma + T$, and $B_{\gamma-1} \subseteq B_\gamma$ guarantees that B_γ satisfies all other requirements as well; for this case, the inductive step is complete.

If $\gamma - 1$ does not exist, then let $B_\gamma = \bigcup (B_\beta : \beta < \gamma)$. The only thing to show is that $G_\gamma = B_\gamma + T$. First, $B_\gamma \cap T = 0$; for, if $t \in B_\gamma \cap T$, then $t \in B_\beta$ for some β preceding γ, and therefore $t \in B_\beta \cap T = 0$. Next, if $g \in G_\gamma$, then, according to (7), $g \in G_\beta$ for some β preceding γ; hence $g = g' + g''$ with $g' \in B_\beta \subseteq B_\gamma$ and $g'' \in T$; so B_γ and T generate G_γ.

Consequently, after σ steps one obtains a subgroup B_σ such that $G = G_\sigma = B_\sigma + T$. This completes the proof of A1.

Proof of A2.[1]) Let T be the maximal torsion subgroup of G, and let $G = A + T$. Then nT is the maximal torsion subgroup of nG; $nG = nA + nT$; and, as the kernel of the endomorphism $g \to ng$ avoids A, $nA \cong A$. Hence H and its subgroup nG satisfy the assumptions of A1, and so one concludes that H splits.

Proof of B1.

Let $H = \sum(A_\alpha : \alpha < \sigma)$ where the A_α are torsion free groups of rank 1, σ is an ordinal, and α runs through all the ordinals which precede σ. Further, assume that $T(A_\alpha) \geqq T(A_\beta)$ whenever $\alpha \leqq \beta < \sigma$.

For each ordinal β such that $\beta \leqq \sigma$, let $H_\beta = \sum(A_\alpha : \alpha < \beta)$, and let G_β be the pure closure of H_β in G. Then, as in the proof of A1, one has that:

(8) if $\beta \leqq \gamma \leqq \sigma$, then $H_\beta \subseteq H_\gamma$ and $G_\beta \subseteq G_\gamma$;

(9) if $\lambda \leqq \sigma$ and λ is a limit ordinal, then

$$H_\lambda = \bigcup (H_\beta : \beta < \lambda) \quad \text{and} \quad G_\lambda = \bigcup (G_\beta : \beta < \lambda);$$

[1]) The author thanks Professor A. Kertész for pointing out oversights in the original versions of the proofs of A2 and B2.

(10) if $\beta < \sigma$, then $H_{\beta+1} = H_\beta + A_\beta$; and

(11) if $\beta \leqq \gamma \leqq \sigma$, then $G_\beta \cap H_\gamma = H_\beta$.

The main step of the proof is the application of a lemma of BAER [1] (Lemma 46.3 in FUCHS [2]). This implies that G_β is a direct summand of $G_{\beta+1}$ for every relevant value of β; the conditions of its application are that

(12) $G_{\beta+1}/G_\beta \cong A_\beta$, and every element of the difference set $G_{\beta+1} - G_\beta$ has type $T(A_\beta)$ in $G_{\beta+1}$, whenever $\beta < \sigma$.

The next task is therefore to prove that (12) is true.

First, one remarks that, because of the purity of $G_{\beta+1}$ in G, the type of an element taken in $G_{\beta+1}$ is the same as that taken in G. Next, the type of a non-zero element h in H is the same as that in G: for, if $h = p^k g$ with $g \in G$, $g \notin H$, then $g + H$ has p-power order in G/H; now, if p^m is the highest power of p dividing n, then $p^m g \in H$; consequently, the p-height of h in G is not greater than m plus the p-height of h in H; as m is 0 for almost all primes p, the assertion follows. Thirdly, as every H_α and every A_α is pure in H, the type of an element of H_α or A_α in H_α or A_α, respectively, is again the same as that in G. Thus there will be no ambiguity if the symbol $T(g)$ will be used for the type of g in any of these subgroups which contains g.

Immediate consequences of these remarks are that $T(a) = T(A_\alpha)$ whenever $0 \neq a \in A_\alpha$, for every α which occurs here, and that, since $T(A_\alpha) \geqq T(A_\beta)$ when $\alpha \leqq \beta < \sigma$, $T(h) \geqq T(A_\beta)$ for each non-zero element h of $H_{\beta+1}$.

Let K be the subgroup generated by G_β and A_β. Since $A_\beta \subseteq H_{\beta+1} \subseteq G_{\beta+1}$, K is contained in $G_{\beta+1}$. Also, $G_\beta \cap A_\beta = G_\beta \cap H_{\beta+1} \cap A_\beta = H_\beta \cap A_\beta = 0$ [according to (11)], so that in fact $K = G_\beta + A_\beta$. Moreover, $nG_{\beta+1} \subseteq nG \subseteq H = H_\sigma$ and so, again by (11),

$$nG_{\beta+1} \subseteq H_\sigma \cap G_{\beta+1} = H_{\beta+1} \leqq K. \quad (13)$$

Thus the situation is that $G_{\beta+1}/G_\beta$ is torsion free (because G_β is pure), and its multiple $n(G_{\beta+1}/G_\beta)$ is contained in the subgroup K/G_β of rank 1. Hence $(G_{\beta+1}/G_\beta)/(K/G_\beta)$ is finite, as can be seen directly or from Lemma 5 of Procházka [3]. This in turn implies that $G_{\beta+1}/G_\beta$ is isomorphic to K/G_β and so to A_β. The first part of (12) is thus proved.

Let g be an arbitrary element of $G_{\beta+1} - G_\beta$. The line (13) gives that $ng \in H_{\beta+1} = H_\beta + A_\beta$, whence $ng = h + a$ with $h \in H_\beta$, $a \in A_\beta$. Since G_β is pure, a cannot be 0. If $h = 0$, then $T(ng) = T(a) = T(A_\beta)$. If $h \neq 0$, the type of ng can still be evaluated with reference to this direct decomposition: as $T(h) \geqq T(A_\beta)$ and $T(a) = T(A_\beta)$, one gets that $T(ng) = T(h) \cap T(a) = T(A_\beta)$. Thus $T(ng)$, which is, of course, the same as $T(g)$, is in either case $T(A_\beta)$. This completes the proof of (12).

Thus it is possible to apply BAER's lemma, and so to conclude that $G_{\beta+1} = G_\beta + B_\beta$ with suitable subgroups B_β, for each β which precedes σ. Note that $B_\beta \cong G_{\beta+1}/G_\beta \cong A_\beta$.

It is easy to see that the B_β are independent. Indeed, a non-trivial relation $b_{\beta_1} +$

$+ \ldots + b_{\beta_k} = 0$ with $b_{\beta_1} \in B_{\beta_1}, \ldots, 0 \neq b_{\beta_k} \in B_{\beta_k}$, $\beta_1 < \beta_2 < \ldots < \beta_k$, would give $b = b_{\beta_1} + \ldots + b_{\beta_{k-1}} \in G_{\beta_k}$, $b + b_{\beta_k} = 0$, contrary to the directness of the sum $G_{\beta_k} + B_{\beta_k}$. Thus the subgroup generated by the B_β is in fact their direct sum.

It remains to prove that this subgroup is the whole of G; for then arbitrary isomorphisms between the A and B induce an isomorphism between G and H. In other words, as $G = G_\sigma$, one requires a special case (that in which $\beta = \sigma$) of the following statement:

$$G_\beta = \sum(B_\alpha : \alpha < \beta) \quad \text{whenever} \quad \beta \leqq \sigma . \tag{14}$$

This statement is conveniently proved by a contradiction argument. Suppose that (14) is not true, and let γ be the first ordinal such that (14) is false for $\beta = \gamma$. Then $\gamma \neq 0$, for (14) is trivially true when $\beta = 0$. If γ has a predecessor, then (14) is true for $\gamma - 1$, and so a contradiction follows: $G_\gamma = G_{\gamma-1} + B_{\gamma-1} = \sum(B_\alpha : \alpha < \gamma - 1) + B_{\gamma-1} = \sum(B_\alpha : \alpha < \gamma)$. If γ is a limit ordinal, then (9) can be applied to deduce that

$$\sum(B_\alpha : \alpha < \gamma) = \bigcup \{\sum(B_\alpha : \alpha < \beta) : \beta < \gamma\} = \bigcup (G_\beta : \beta < \gamma) = G_\gamma ,$$

which is again a contradiction. This completes the proof of B1.

Proof of B2. As $g \to ng$ is an isomorphism of G onto nG, one has that $G \cong nG$ and that B1 can be applied to H and nG in place of G and H. Hence B1 gives that $nG \cong H$, and so one concludes that $G \cong H$.

The example. Let R_i be the additive group of those rational numbers whose denominators are composed of the first i odd primes only, for $1 \leqq i < \omega$. Let $(u_i : 1 \leqq i < \omega)$ be an independent subset of a torsion free divisible abelian group, and let U be the subgroup given as $\sum(R_i u_i : 1 \leqq i < \omega)$. It is immediate that the type of a non-zero element u in U is $T(R_j)$ where j is the first suffix for which the $R_j u_j$-component of u (in this direct decomposition of U) is not 0. Consequently, the set of types of elements of U is $(T(R_i) : 1 \leqq i < \omega)$, a naturally well-ordered set. Moreover, the subgroup U_j consisting of the elements of type at least $T(R_j)$ is precisely $\sum(R_i u_i : j \leqq i < \omega)$.

Let V be the subgroup generated by $2U$ and the elements $u_i - u_{i+1}$, for $1 \leqq i < \omega$. It is easy to see that the type of a non-zero element of V taken in V is the same as that taken in U.

Assume, for a contradiction argument, that V is a direct sum of groups of rank 1, $V = \sum(A_i : 1 \leqq i < \omega)$; here the same suffixes can be used as before, since V is of countable rank, but there is no relation implied between A_i and $R_i u_i$. The only thing that follows immediately is that to each i there is an i' such that $T(A_i) = T(R_{i'})$. As $2u_1 \in V$, $2u_1 \in \sum(A_i : 1 \leqq i \leqq k)$ for some positive integer k. Let $j = 1 + \max(i' : 1 \leqq i \leqq k)$. Then in the direct decomposition

$$V = \sum(A_i : 1 \leqq i' < j) + \sum(A_i : j \leqq i' < \omega) . \tag{15}$$

$2u_1$ is contained in the first summand, while the second summand is precisely the subgroup V_j consisting of those elements of V whose types are at least $T(R_j)$, so that this second summand contains $2u_j$. If the corresponding decomposition of $(u_1 - u_2) + \ldots + (u_{j-1} - u_j)$ is $a + b$, then $2a + 2b = 2u_1 - 2u_j$, whence $2a = 2u_1$ with $a \in V$, and then of course $u_1 = a \in V$ too. It follows therefore that u_1 can be written as a linear combination of finitely many of the generators of V; that is,

$$u_1 = n_1(u_1 - u_2) + \ldots + n_m(u_m - u_{m+1}) + 2r_1u_1 + \ldots + 2r_{m+1}u_{m+1}$$

for suitable integers $n_1, \ldots, n_m$ and elements r_i of the R_i, $i = 1, \ldots, m + 1$. Comparing the coefficients of the independent elements $u_1, \ldots, u_{m+1}$ on the two sides, one obtains that

(16.1) $$1 = n_1 + 2r_1 \,,$$
(16.2) $$0 = -n_1 + n_2 + 2r_2 \,,$$
$$\vdots$$
(16. m) $$0 = -n_{m-1} + n_m + 2r_m \,,$$
(16. $m + 1$) $$0 = -n_m + 2r_{m+1} .$$

Since the denominators of the r_i are divisible by odd primes only, (16.1)–(16. $m + 1$) imply that the r_i are in fact integers. Thus (16. $m + 1$) implies that n_m is even; hence (16. m) implies that n_{m-1} is even; and so on: (16.2) gives that n_1 is even, so that (16.1) represents 1 as the sum of two even numbers, which is impossible. This contradiction proves that V cannot be a direct sum of groups of rank 1.

References

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[3] *L. Procházka:* Zur Spaltbarkeit von gewissen abelschen Erweiterungen der spaltbaren abelschen Gruppen. Czechosl. Math. J. *11* (*86*) (1961), 365–380. (Russian, with German summary.)

Резюме

ОБ ОДНОЙ СТАТЬЕ ЛАДИСЛАВА ПРОХАЗКИ

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В статье доказаны прежде всего следующие две теоремы.

Теорема А. *Пусть G — абелева группа, пусть n — натуральное число и пусть H — такая подгруппа в G, что $nG \subseteq H$.*

A1. *Если H расщепляема и если ее слагаемое без кручения является прямой суммой групп конечного ранга, то G будет также расщепляемой.*

A2. *Если G расщепляема и если ее слагаемое без кручения является прямой суммой групп конечного ранга, то H будет также расщепляемой.*

Теорема B. *Пусть G — абелева группа без кручения, пусть n — натуральное число и пусть H — такая подгруппа в G, что $nG \subseteq H$.*

B1. *Если H является прямой суммой групп ранга 1, типы которых представляют инверсно вполне упорядоченное множество относительно частичного упорядочения типов, то $H \cong G$.*

B2. *Если G является прямой суммой групп ранга 1, типы которых представляют инверсно вполне упорядоченное множество относительно частичного упорядочения типов, то $G \cong H$.*

Кроме того, в статье построен пример группы U, являющейся прямой суммой групп без кручения ранга 1, типы которых представляют вполне упорядоченное множество относительно частичного упорядочения типов; но, группа U содержит такую подгруппу V, что $2U \subseteq V$ и подгруппа V не является прямой суммой групп ранга 1.