Direct complementation in groups with operators

To Professor REINHOLD BAER on his 60th birthday

By

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1. One of the starting points of the theory of ordinary representations is Maschke's Theorem, which states that every representation of a finite group over a field whose characteristic does not divide the order of the group is completely reducible (see e.g. VAN DER WAERDEN [4], p. 182). The main step in the proof of this theorem is of much wider application. In the theory of groups with operators (in the sense of KUROSH [3], § 15) it can be used to prove a result on direct complementation: If A is a group and G a finite group of operators of A, then every G-admissible direct factor of A whose centre is in a sense prime to the order of G has a G-admissible direct complement in A (Lemma 2.1 and Theorem 2.2).

Another investigation [2] led us to ask whether the restriction that G be finite could be weakened. It is easy to construct examples (which we, therefore, omit) to show that this is unlikely unless there is some compensating strengthening of or addition to the other conditions of the theorem. We present a result in this direction: roughly speaking, it says that, for direct factors whose centres are in a sense small, the same conclusion holds whenever the group of operators is periodic with finite central factor group (Theorem 3.2).

The paper is concluded with an example due to B. H. NEUMANN. This shows the falsity of the proposition which is obtained from Theorem 3.2 when "periodic with finite central factor group" is replaced by "countable, metabelian, and of exponent 4". There remains a considerable gap between Theorem 3.2 and this negative result. For instance, we do not know what happens if the replacement is chosen as "nilpotent of class 2, and of finite exponent": is the statement so obtained true or false?

We thank Professor NEUMANN for his example.

2. In this and in the following section, we use additive notation in "ordinary" groups and multiplicative notation in groups of operators. If S is a set of operators of a group A, if $s \in S$ and $a \in A$, the composition of a and s is written as as.

Let Π be a set of prime numbers. As usual, a group is called a *Π-group* if it is periodic and if each prime which occurs as the order of an element of this group belongs to Π. We call a group *n-aperiodic* if it has no element (other than the identity) whose order is a divisor of n.

In order to ease the formulation of the first result of this paper, we do it in two parts: first, a simple lemma which reduces the problem to the abelian case, and then, the theorem dealing with this case.

2.1. Lemma. *Let A be a group and G a group of operators of A; let D be a G-admissible subgroup which is a direct summand of A (qua group without operators); let Z be the centralizer of D in A, and denote $Z \cap D$ by C. Then Z, C, Z' (the commutator subgroup of Z), Z/Z', $(C + Z')/Z'$ are all G-admissible; $C \cap Z' = 0$; $(C + Z')/Z'$ is a direct summand of Z/Z'; and D has a G-admissible direct complement in A if and only if $(C + Z')/Z'$ has one in Z/Z'.*

The proof of this is quite straightforward, and we omit it.

2.2. Theorem. *Let A be an abelian group and G a finite group of operators of A; let the order of G be mn; let D be a G-admissible subgroup which is a direct summand of A (qua group without operators); and let A be divisible by m, D divisible also by n, A/D m-aperiodic, and D n-aperiodic. Then D has a G-admissible direct complement in A.*

Proof. Let $A = B + D$ be a direct decomposition and $\delta : A \to D$ the corresponding projection onto D (i.e., if $a \in A$, $a = b + d$, $b \in B$, $d \in D$, let $a\delta = d$). Define a mapping $\pi : A \to D$ by postulating that $n(a\pi) = \sum (ag^{-1}\delta g : g \in G)$. Such an element $a\pi$ always exists in D, for the right hand side is in D and D is divisible by n. Also, for any two elements x, y of the n-aperiodic abelian group D, $nx = ny$ implies that $x = y$; so $a\pi$ is uniquely determined by this formula. It is easy to check that π is an endomorphism of A. Moreover, if h is an arbitrary element of G,

$$n((ah)\pi) = \sum (ahg^{-1}\delta g : g \in G) = \sum (a(gh^{-1})^{-1}\delta(gh^{-1}) : gh^{-1} \in G)h = n(a\pi)h;$$

for, as g runs through the elements of G, so does gh^{-1}, and vice versa. Thus $(ah)\pi = (a\pi)h$ for every h in G, and so the kernel K of π is G-admissible. If $d \in D$, then $n(d\pi) = \sum (d : g \in G) = mnd$, so that $d\pi = md$. Hence $D\pi = mD = D = A\pi$, and $K \cap D = \text{set}\,(d \in D : md = 0)$. From this it follows that $A = D + K$; for if $a\pi = d\pi$, then $a - d \in K$. Now $K/(K \cap D)$, being isomorphic to $(D + K)/D$ and so to A/D, is m-aperiodic, while the order of each element of $K \cap D$ divides m; hence $K \cap D$ is a bounded pure subgroup of K. It follows now from a theorem of Kulikov (Theorem 24.5 in Fuchs [1]) that $K \cap D$ is a direct summand of K; let

$$K = (K \cap D) + D^*$$

be a direct decomposition. Then also $A = D + D^*$ and $D \cap D^* = 0$; consequently D^* is isomorphic to A/D and so it is divisible by m. Thus

$$mK = m(K \cap D) + mD^* = mD^* = D^*$$

which shows that D^* is a characteristic subgroup of the G-admissible K, so that D^* is also G-admissible — in D^* we have the required complement for D.

2.3. Remark. *If I is a set of operators of A, if D and B are both I-admissible, and if $axg = agx$ for every a in A, x in I, and g in G, then D^* is also I-admissible.*

This follows immediately from the way D^* was chosen: one notes first that in this case $(a\delta)x = (ax)\delta$ and hence $(a\pi)x = (ax)\pi$ for every a in A and x in I; thus K is I-admissible, and therefore mK, or D^*, is also I-admissible.

2.4. Corollary. *If an abelian group A is divisible by all the primes which occur as orders of elements in this group, then its maximal periodic subgroup D has an admissible direct complement in A with respect to every finite group G of operators.*

(Of course, this complement need not be the same for every such G.) This follows as D is divisible and hence, according to a theorem of BAER (Theorem 18.1 in FUCHS [1]), it is a direct summand of A, and n can be chosen as the largest divisor of the order of G for which A is n-aperiodic.

3. A group is called *G-monolithic* if G is a set of operators of this group and if the intersection of all the non-trivial G-admissible normal subgroups of the group is non-trivial; this intersection is then the *G-monolith* of the group.

3.1. Lemma. *Let C be a G-monolithic abelian group, and suppose that G is a Π-group with finite central factor group, while the G-monolith M of C is not a Π-group. Then C is the direct sum of isomorphic, locally cyclic, p-groups for some prime p not in Π, and only the characteristic subgroups of C are G-admissible.*

Proof. (a) First, let us discuss the structure of M. If N is a characteristic subgroup of M, then N is also a G-admissible subgroup of C and so, according to the definition of M as G-monolith, N is either trivial or it contains and hence equals M. Thus M is a characteristically simple abelian group, i.e., either an aperiodic divisible abelian group or an elementary abelian p-group for some prime p (which, by our assumption that M is not a Π-group, cannot be in Π). We eliminate the first possibility as follows. Suppose that M is aperiodic and x is a non-trivial element of M. The subgroup generated by the elements xg where g runs through G is a G-admissible non-trivial subgroup of C; hence it is M. As in this case M is divisible, it has an element x_1 such that $2x_1 = x$; by what has been said, this x_1 must be expressible as $\sum n_g xg$ where g runs through some finite subset S_x of G and the n_g are integers. For $i \geqq 1$, define x_{i+1} inductively by $x_{i+1} = \sum (n_g x_i g : g \in S_x)$. It is easy to see, by a straightforward induction on i, that $2x_{i+1} = x_i$ for $i = 1, 2, \ldots$. Now the subgroup generated by the x_i is abelian and divisible by 2; as such, it cannot be finitely generated; on the other hand, it is contained in the abelian subgroup generated by the finitely many elements xg with $g \in S_x$. This contradiction completes the elimination; so we conclude that M is an elementary abelian p-group for some prime p not in Π.

(b) We have to introduce some further terms. Let it be recalled that every element g of G induces an automorphism $c \to cg$ on C. The automorphisms induced by the elements of a subgroup H of G generate a subring in the ring of endomorphisms of C. We call this subring the *associate* of H, and consider it a ring of operators rather than of endomorphisms. Note that a subgroup of C is H-admissible if and only if it is admissible with respect to the associate of H. The associate of G will be denoted by R, and the associate of the centre I of G by S; clearly, S is in the centre of R.

(c) As G is periodic and its central factor group is finite, it follows that G is locally finite; moreover, there is a finite subgroup F in G which together with I generates G. Also, each element r of R is contained in the associate of some finite subgroup G_r of G.

(d) If q is a prime other than p, the maximal q-subgroup of C is a characteristic and therefore G-admissible subgroup which avoids M, hence it must be trivial. Thus the maximal periodic subgroup P of C is a p-group.

(e) Let L be the lowest layer of P, i.e., the subgroup consisting of the elements whose orders divide p; then M is contained in L. Suppose that $Ms = 0$ but $Ls \neq 0$ for some element s of S. Clearly, Ls is R-admissible and so it contains M. Let G_s be a finite subgroup of G whose associate contains s. By Theorem 2.2, M has a G_s-admissible direct complement M^* in L. Since $Ms = 0$ and $L = M + M^*$, it follows that $Ls = M^*s$, and so $M^* \geqq M^*s = Ls \geqq M$. This is a contradiction; therefore $Ms = 0$ implies that $Ls = 0$.

(f) Let $0 \neq x \in L$; we claim that in this case xS is a minimal S-admissible subgroup. The proof runs as follows. Let T be the ideal of S which consists of the annihilators of M in S; then $pS \subseteqq T$. If s is an element of S but not of T, then the kernel of s does not contain M though it is R-admissible, hence this kernel must be trivial. The subgroup Ms is non-trivial and also R-admissible, so it must equal M. Consequently s induces an automorphism on M. Hence the multiplicative semigroup of (the non-zero elements of) S/T is faithfully represented in the automorphism group of M. Now s is contained in the associate U of some finite subgroup G_s of the centre I of G; this U, and hence also $U/(U \cap T)$, is additively generated by finitely many elements; moreover, $p(U/(U \cap T)) = 0$, so that $U/(U \cap T)$ is finite. Thus the multiplicative semigroup of $(U + T)/T$ is faithfully represented by a finite sub-semigroup of the automorphism group of M, i.e., by a subgroup of this group, and so s has an inverse s' modulo T: $ss' = 1 + t$ where 1 is the identity of S and $t \in T$. Now if xs is a non-zero element of xS, then $Ls \neq 0$ and hence, according to (e), $Ms \neq 0$, that is, $s \notin T$. Thus we are in the situation discussed above: there is an s' in S and a t in T such that $ss' = 1 + t$. Again by (e), $Mt = 0$ implies that $Lt = 0$. So $xss' = x$, and this shows that the smallest S-admissible subgroup which contains xs (i.e., xsS) will also contain x and hence equal xS. As xs was arbitrary, it follows that xS is a minimal S-admissible subgroup.

(g) We are ready to prove that $M = L$. Let N be a subgroup of L maximal with respect to avoiding M and being S-admissible. If $x \in L$, $x \notin N$, then $xS \nleqq N$ and so, according to (f), $xS \cap N = 0$. On the other hand, by the maximality of N, the subgroup generated by N and xS intersects M non-trivially: we have $0 \neq xs + y = u \in M$ for some $s \in S$ and $y \in N$. Thus $xs = u - y \in M + N$ and $xs \neq 0$ (for $M \cap N = 0$); so $xS \cap (M + N) \neq 0$. Another reference to (f) confirms now that $x \in xS \leqq M + N$, and so it follows that $L = M + N$. We have shown that M has an S-admissible, i.e., I-admissible, direct complement N in L. Hence Theorem 2.2 with its amendment 2.3 implies that N can be chosen to be also F-admissible; as I and F generate G, this N will be a G-admissible subgroup which avoids M and must therefore be trivial. Hence indeed $L = M + N = M$.

(h) The structure of P is now easily described. The elements of M whose height in P (see e.g. Fuchs [1], p. 16) is at least k where k is a non-negative integer or infinity form a characteristic and therefore G-admissible subgroup in C; hence all the non-zero elements of M are of the same height in P. If this common height is infinity, then P is divisible and so a direct product of groups of type $C(p^\infty)$; while if it is a finite number $k-1$, then P is the direct product of cyclic groups of order p^k. For every non-negative integer i, let P_i be the subgroup of P generated by (and consisting of) the elements of order at most p^i. These and P are all the characteristic subgroups of P, and they are also the only G-admissible ones: for, if B were another G-admissible subgroup of P, there would be an integer i such that $P_i < B$ but $P_{i+1} \lneqq B$, and then $p^i B \cap M$ would be a proper non-trivial G-admissible subgroup in M, and this is impossible.

(i) It follows that P is a direct summand of C; for P is either a divisible or a bounded pure subgroup of C, and so either a theorem of Baer or one of Kulikov (Theorem 18.1 or 24.5 in Fuchs [1]) applies.

(j) If M is in fact an S-monolith of C, then $P = C$ and so the conclusion of the lemma holds. In order to see this, assume that M is the S-monolith of C and consider an arbitrary non-zero element c of C; we have to show that $c \in P$. As cS is a non-trivial S-admissible subgroup, $M \leqq cS$ and so $0 \neq cs \in M$ for some s in S. Suppose that $Ps = 0$. Let G_s be a finite subgroup of G such that s is contained in the associate of G_s; then (i) and Theorem 2.2 imply that P has a G_s-admissible direct complement P^* in C. Then $cs \in Cs = (P + P^*)s = P^*s \leqq P^*$ follows, contrary to $0 \neq cs \in$ $\in M \leqq P$ and $P \cap P^* = 0$. Hence $Ps \neq 0$, so that the periodic part of the kernel K of s is a proper R-admissible subgroup of P, say it is P_i. Then $p^i K$ is an R-admissible subgroup which, being aperiodic, avoids M and therefore must be trivial. It follows that $K \leqq P$. Moreover, Ps is a non-trivial R-admissible subgroup, so it contains M and a fortiori cs. If now $y \in P$ and $ys = cs$, then $y - c \in K \leqq P$ implies that $c \in P$, as required.

(k) The G-monolith M is the direct sum of finitely many minimal S-admissible subgroups $M_1, \ldots, M_k$. The key to this step is (f). Let x be an arbitrary non-zero element of M, and let N be the subgroup generated by the subgroups xfS for all f in F. Then N is I-admissible and also F-admissible, so it is G-admissible and hence equal to M. According to (f), the xfS are minimal S-admissible subgroups. If F_0 is a subset of F maximal with respect to the sum $\sum (xfS : f \in F_0)$ being direct, then

$$xf'S \cap \sum (xfS : f \in F_0)$$

must be non-trivial for each f' in F/F_0; this intersection is also an S-admissible subgroup of the minimal S-admissible subgroup $xf'S$, so

$$xf'S \leqq \sum (xfS : f \in F_0)$$

for every f' in F, and the result follows.

(l) It is easy to dispose of the case when P has finite exponent, say $P = P_i$. For then $p^i C$ is a G-admissible subgroup which, being aperiodic, avoids M; so we deduce that $p^i C = 0$, whence $C = P$, and so C has the claimed structure.

(m) So we may assume that P is divisible. Let C_0 be C, and define C_i for $i = 1, \ldots, k$

inductively as a subgroup of C_{i-1} maximal with respect to being S-admissible, avoiding M_i, and containing $M_{i+1} + \cdots + M_k$ (if $i < k$). It is easy to check that each C_{i-1}/C_i is S-monolithic and its S-monolith is $(M_i + C_i)/C_i$. Hence (g) and (j) apply to each of these factors (with I in place of G): the C_{i-1}/C_i are p-groups and their lowest layers are the $(M_i + C_i)/C_i$. It follows that C/C_k is a p-group and its lowest layer is $(M + C_k)/C_k$. As C_k avoids M, it avoids P. The divisible group $(P + C_k)/C_k$ is contained in the p-group C/C_k and their lowest layers are the same, hence

$$(P + C_k)/C_k = C/C_k,$$

and so $P + C_k = C$. This means that P has an S-admissible, i.e., I-admissible direct complement C_k in C. We invoke again Theorem 2.2 and Remark 2.3 to prove that C_k can be chosen to be also F-admissible and hence G-admissible. However, it will always avoid M, and so it must be trivial. Thus indeed $C = P + C_k = P$. According to (h), this completes the proof of the lemma.

3.2. Theorem. *Let A be a group and G a group of operators of A; suppose that G is a Π-group with finite central factor group; and let D be a G-admissible subgroup and a direct summand of A, such that the centre C of D is either trivial or has a G-monolith which is not a Π-group. Then D has a G-admissible direct complement in A.*

Proof. Lemma 2.1 allows us to restrict our attention to the case when A is abelian and C, that is now D, is G-monolithic. The structure of C is given to us by Lemma 3.1. Let B be a subgroup of A maximal with respect to being G-admissible, avoiding C, and containing $p^i A$ if C has finite exponent p^i. Let us consider A/B; it will be convenient to denote this factor group by $\bar{A}$, and in general to use the bar to denote images under the natural homomorphism of A onto $\bar{A}$. Every non-trivial G-admissible subgroup of $\bar{A}$ has non-trivial intersection with $\bar{C}$ and hence contains the G-monolith $\bar{M}$ of $\bar{C}$, so that $\bar{M}$ is the G-monolith of $\bar{A}$. Thus Lemma 3.1 can be applied to $\bar{A}$ (in place of C); we obtain from this that $\bar{A}$ is a p-group and its lowest layer is $\bar{M}$. Now $\bar{A}$ contains the subgroup $\bar{C}$ which has the same lowest layer as $\bar{A}$; moreover, $\bar{C}$ is either divisible or the direct sum of cyclic groups of order p^k; in the second case the exponent of $\bar{A}$ is also p^k; hence, in either case, it follows that $\bar{C} = \bar{A}$. Thus $A = B + C$, and this sum is direct.

4. We conclude the paper with the example promised in the introduction.

In this section it is more convenient to use multiplicative notation in all groups, and correspondingly to denote the composition of an element a (of an "ordinary" group) and an operator s as a^s.

Let $X = gp(w, x; w^2 = x^3 = 1, x^w = x^2)$ and $Y = gp(y_1, y_2, \ldots; y_i^2 = 1, y_i y_j = y_j y_i)$; X is a symmetric group of degree 3 and Y is a countably infinite group of exponent 2. The unrestricted direct power $X \operatorname{Dp} Y$ of X indexed by Y is the group whose elements are the functions $f: y \to f(y)$ from Y to X and whose multiplication is defined by $(ff')(y) = f(y)f'(y)$ for every y in Y. The support of an element f of $X \operatorname{Dp} Y$ is the subset of Y consisting of all those elements on which the value of f is different from 1. The restricted direct power $X \operatorname{dp} Y$ is a subgroup of $X \operatorname{Dp} Y$, namely the one formed by the elements of finite support. If $z \in Y$ and $f \in X \operatorname{Dp} Y$, let f^z be the element of

$X \operatorname{Dp} Y$ defined by $f^z(y) = f(yz^{-1})$ for every y in Y. The set of all formal products yf where $y \in Y$ and $f \in X \operatorname{Dp} Y$ is a group, called the unrestricted abstract wreath product $X \operatorname{Wr} Y$ of X and Y, under the multiplication $(yf)(y'f') = (yy')(f^{y'}f')$. The corresponding restricted abstract wreath product $X \operatorname{wr} Y$ is the subgroup of $X \operatorname{Wr} Y$ which consists of the yf with $f \in X \operatorname{dp} Y$.

Let B be the subgroup of $X \operatorname{Wr} Y$ generated by $X \operatorname{wr} Y$ and the element $1 f_0$ which is such that $f_0(y) = x$ for every y in Y. It is easy to verify that B has a normal Sylow 3-subgroup A, and that the Sylow 3-subgroup D of $X \operatorname{wr} Y$ is contained in every non-trivial normal subgroup of B. As A is elementary abelian, D is a direct factor of A; and of course D is normal in B. The factor group B/A is countable, metabelian, and has exponent 4. Let this group be called G, and make it into a group of operators of A by defining a^{Ab} as $b^{-1}ab$; as A is abelian, this definition is independent of the choice of the representative b within the coset Ab. It follows now that D is the G-monolith of A; hence D is itself G-monolithic, and it cannot have a G-admissible complement in A.

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