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# PUBLICATIONES MATHEMATICAE

TOMUS 7.

DEBRECEN
1960



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# Rings covered by minimal left ideals.

To Professor O. Varga on his 50th birthday.

By L. G. KOVÁCS (Debrecen) and J. SZÉP (Szeged).

Several results are known concerning coverings of groups and semigroups, e. g. form papers of A. H. CLIFFORD, P. G. KONTOROVIC, G. A. MILLER, B. H. NEUMANN, S. SCHWARZ, M. SUZUKI, J. SZÉP, M. TAKAHASI, J. W. YOUNG etc. In this short note we present the solution of a similar problem for rings, namely we describe all (associative) rings that are covered by (in other words, are the set-theoretic unions of) their minimal left ideals. Since every ring can be considered as a (left) module over itself and then the left ideals are the submodules, it is relevant to give all modules that are covered by minimal submodules. We do this first, and use the result in the solution of our problem on rings. — By modules we always mean left modules, but the whole paper can obviously be dualized for right modules and rings covered by minimal right ideals.

Our terminology follows that of A. KERTÉSZ [1]. If $R$ is a ring, $R^*$ stands for the Dorroh-extension of $R$, that is for the ring of all pairs $\langle r, n\rangle$ (with $r\in R$, $n$ rational integer), addition being defined component-wise and multiplication by

$$\langle r, n\rangle\langle r', n'\rangle = \langle rr' + nr' + n'r, nn'\rangle .$$

The subset of pairs of the form $\langle r, 0\rangle$ is naturally identified with $R$ and is an ideal of $R^*$, while $\langle 0, 1\rangle$ serves as unit element. We consider any (left) $R$-module $G$ as a module over $R^*$, writing

$$\langle r, n\rangle g = rg + ng$$

for every element $g$ of $G$. The order $O(g)$ of $g$ is the left ideal formed by the elements of $R^*$ for which $\langle r, n\rangle g = 0$; $O(G)$ is the intersection of the orders of all elements of $G$, and is in fact a two-sided ideal. If $L$ is any left ideal of $R^*$, we write $L^+$ for $L$ considered as an $R^*$-module. The cyclic module $\{g\} = R^*g$ generated by $g$ is known to be isomorphic with $R^{*+}/O(g)^+$. We shall often make use of the following elementary lemma.

Lemma (T. Szele [2]). *If a ring has no proper left ideals then it is either a skew field or a zero-ring of prime order.*[1])

**Theorem 1.** *An R-module G is covered by its minimal submodules if and only if either $R^*/O(G)$ is a skew field or G is itself a simple*[2]) *module.*

Proof. If $G$ is non-simple and is covered by minimal submodules then any nonzero element $g$ of $G$ generates a proper minimal submodule. Take any element $g'$ outside $\{g\}$ and any $\langle r, n\rangle$ in $O(g')$. Since $(g'+g)\notin\{g\}$ and $\{g'+g\}$ is minimal, $\{g'+g\}\cap\{g\}=0$; so

$$\langle r, n\rangle(g'+g)=\langle r, n\rangle g\in\{g'+g\}\cap\{g\}$$

shows that $\langle r, n\rangle\in O(g)$. Thus we have $O(g')\subseteq O(g)$; by symmetry also $O(g)\subseteq O(g')$, so in fact $O(g')=O(g)$ for every $g'$ outside $\{g\}$. Similarly $O(g'')=O(g')=O(g)$ for every $g''$ $(\in G)$ outside $\{g'\}$. As $g\notin\{g'\}$ and $\{g\}$ is minimal, any nonzero element of $\{g\}$ is outside $\{g'\}$, so it has the same order as $g'$ and $g$. Consequently every nonzero element of $G$ has the same order as $g$; this yields $O(g)=O(G)$. Now $\langle 0, 1\rangle+O(G)$ is a unit element in $R^*/O(G)$; this factor-ring has no proper left ideals since otherwise $R^{*+}/O(g)^+\simeq\{g\}$ would have proper submodules; so the above Lemma applies to prove that $R^*/O(G)$ is a skew field.

Conversely, assume that $R^*/O(G)$ is a skew field. This means that $G$ is essentially a vector space (over this skew field); as such, $G$ is covered by one-dimensional subspaces, that is by minimal submodules.

As a corollary to Theorem 1 we obtain that

*an abelian group G is covered by minimal subgroups if and only if it is an elementary p-group.*

Indeed, $G$ is a module over the ring consisting of one element; in this case $R^*$ is the ring of rational integers, its only homomorphic images that are skew fields are the finite prime fields, so every element of $G$ has to be of (the same) prime order. — Of course, this corollary can easily be deduced also directly.

**Theorem 2.** *A ring R is covered by its minimal left ideals if and only if*

*either R is a zero-ring on an elementary p-group,*

*or R has a left unit e, the left ideal $R^*e=Re$ generated by e is a skew field, and the left annihilators of R together with Re generate the whole ring R.*

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1) A zero-ring is a ring in which every products 0.

2) A module is called simple if it has no proper submodules.

REMARK. All rings of the type mentioned second in Theorem 2 can be constructed in the following way. Take a skew field $F$ and a left vector space $V$ over $F$. Define $R$ to be the ring of all pairs $\langle f, v\rangle$ $(f \in F, v \in V)$ with component-wise addition and $\langle f, v\rangle\langle f', v'\rangle = \langle ff', fv'\rangle$ as multiplication. $F$ and the dimension $n$ of $V$ form a complete set of invariants of $R$.

PROOF. Assume that $R$ is covered by its minimal left ideals; then $R^+$ is covered by minimal submodules and so Theorem 1 applies to $G = R^+$. If $R^+$ is simple then $R$ has no proper left ideals and an application of the Lemma finishes the proof of the direct part of the theorem. If $R^+$ is non-simple then $R^*/O(R^+)$ is a skew field. For a zero-ring $R$ this gives at once that the additive group of $R$ is an elementary $p$-group, since $O(R^+) \supseteq R$ and so $R^*/O(R^+)$ is a homomorphic image of the ring of the rational integers.

The remaining case is when $O(R^+)$ does not contain $R$. If so, then $\{R, O(R^+)\}/O(R^+)$ is a nonzero ideal in the skew field $R^*/O(R^+)$, so it must be the whole of $R^*/O(R^+)$; this means that $\{R, O(R^+)\} = R^*$. Let us denote $R \cap O(R^+)$ by $K$; the isomorphism theorem gives that $R^*/O(R^+) \simeq R/K$, so $R/K$ is also a skew field. Further, we know that $O(R^+)$ is a maximal left ideal in $R^*$.

Take any element $e$ from the coset that serves as unit element in $R/K$; we show that $e$ is a left unit in $R$. First of all $e^2 + K = (e+K)^2 = e + K$, so $e^2 - e \in K$; then $e(er-r) = (e^2-e)r = 0$ shows that $e \in O(er-r)$; as $O(er-r)$ contains the maximal left ideal $O(R^+)$ and also $e \notin O(R^+)$ it follows that $O(er-r) = R^*$; in particular, $er - r = \langle 0, 1\rangle(er-r) = 0$, so $er = r$ for any $r$ in $R$.

Next, the left ideal $R^*e = Re$ generated by $e$ is minimal; as $e$ ist not contained in $K$ we have $Re \cap K = 0$. On the other hand, $K$ is a maximal left ideal in $R$ ($R/K$ being a skew field), so $\{Re, K\}$ must be the whole of $R$. Thus any $r$ $(\in R)$ has a unique representation as a sum of an element of $K$ and one of $Re$, which implies that $Re \simeq R/K$. We have now proved the direct part: $e$ is a left unit, $Re$ is a skew field, $K$ consists of left annihilators of $R$, and $R = \{K, Re\}$.

The converse part of the theorem is almost obvious. The case of a zero-ring on an elementary $p$-group is clear. If $e$ is a left unit, $Re$ a skew field, $K = R \cap O(R^+)$, $R = \{K, Re\}$, then $R^+$ is essentially a vector space over $Re$ and as such it is covered by one-dimensional subspaces, that is by minimal submodules; so $R$ is covered by minimal left ideals. This completes the proof.

There is no need to give in detail the application of Theorem 2 to rings covered by left ideals of prime order; one gets the complete description of these rings at once.[3]) To conclude, we only wish to mention another similar result.

*A ring $R$ is covered by minimal (two-sided) ideals if and only if $R$ is either a simple ring or a zero-ring on an elementary p-group.*

If $R$ is non-simple and is covered by the minimal ideals $A_\nu$ $(\nu \in \Gamma)$ then, by the minimality, $A_\nu \cap A_\mu = 0$ and so $a_\nu a_\mu = 0$ whenever $a_\nu \in A_\nu$, $a_\mu \in A_\mu$, $\nu, \mu \in \Gamma$, $\nu \neq \mu$. Further, $a_\nu a'_\nu = (a_\nu + a_\mu) a'_\nu = 0$ if $a_\nu, a'_\nu \in A_\nu$, $0 \neq a_\mu \in A_\mu$, since in this case $a_\nu + a_\mu = a_\lambda \in A_\lambda$ for some $\lambda (\in \Gamma)$ other than $\nu$. Consequently every product in $R$ is 0, $R$ is a zero-ring. But then every subgroup of the additive group $R_+$ of $R$ is the additive group of an ideal, so $R_+$ is covered by minimal subgroups and then the corollary to Theorem 1 proves that $R_+$ is an elementary $p$-group.

## Bibliography.

[1] A. Kertész, Beiträge zur Theorie der Operatormoduln, *Acta Math. Acad. Sci. Hungar.* **8** (1957), 235—257.

[2] T. Szele, Die Ringe ohne Linksideale, *Buletin Stiintific* **1** (1950), 783—789.

*(Received February 19, 1959.)*

[3]) A left ideal is of prime order if its cardinality is a (finite) prime number.